Sehitoglu - Jiang - Stress Invariance

Strain Invariance \( \varepsilon_{ij} = \varepsilon_{ij}^e \)

Stress Invariance \( \sigma_{ij} = \sigma_{ij}^e \)

Then, relax to satisfy the boundary conditions.

Jiang - Sehitoglu coordinate system

After one rolling pass, we must satisfy

\[
\begin{align*}
\varepsilon_{x_r} &= 0 \\
\varepsilon_{y_r} &= 0 \\
\varepsilon_{3_r} &= 0 \\
\sigma_{x_r} &= f_1(\varepsilon) \\
\sigma_{y_r} &= f_2(\varepsilon) \\
\varepsilon_{3_r} &= f_3(\varepsilon) \\
\sigma_{x_2} &= f_4(\varepsilon)
\end{align*}
\]
So, we need to enforce

\[ \Delta \varepsilon_x = \frac{\varepsilon_x b}{M} \]

\[ \Delta \varepsilon_y = 0 \]

\[ \Delta \sigma_z = \sigma_z b \]

\[ \Delta \tau_{xz} = \frac{\tau_{xz} b}{M} \]

where \( \Delta \) denotes the finite increment (\( M \approx 10-20 \))

Plane strain requires \( \varepsilon_y = 0 \) \( \Rightarrow \Delta \varepsilon_y = 0 \)

See equations 4, 5, 13 (Jiang + Sehitoglu)

Incompressibility requires,

\[ \dot{\varepsilon}_x^p + \dot{\varepsilon}_y^p + \dot{\varepsilon}_z^p = 0 \]

(2)

These two equations (1) & (2) provide sufficient # \( \varepsilon \) equations to solve for \( \Delta \sigma_x \) & \( \Delta \sigma_y \).
Background:

Note the constitutive equation is

$$\varepsilon_{ij} = \frac{1}{h} \langle S_{kl} n_{kl} \rangle_{ij}$$

where $\langle \rangle$ is the MacCormack Bracket, that is $\langle x \rangle = 0.5 (x + |x|)$.

$$S_{ij} = \delta_{ij} - \frac{1}{3} S_{ij} \delta_{kk}$$

$$n_{ij} = \frac{S_{ij} - \delta_{ij}}{\sqrt{2} k}$$

$$\alpha_{ij} = \langle S_{kl} n_{kl} \rangle_{ij}$$

is called linear kinematic hardening

where $f = \frac{1}{2} (S_{ij} - \delta_{ij}) (S_{ij} - \delta_{ij}) - \frac{k^2}{2}$

$\alpha_{ij}$ represents the center of yield surface, $k$ is yield stress in shear.
We consider
\[ \varepsilon_{ij}^p = \frac{1}{3} \left( \sigma_{kl} n_{kl} \right)_{ij} \]

Because we are working with Cartesian coordinates, we note that
\[ \sigma_{kl} n_{kl} = \delta_{kl} n_{kl} \] noting that \( n_{kk} = 0 \). Proof:

\[ \left( \sigma_{kl} - \frac{1}{3} \delta_{kl} \sigma_{qq} \right) n_{kl} \]

\[ \sigma_{kl} n_{kl} = \delta_{kl} n_{kl} - \frac{1}{3} n_{kk} \sigma_{qq} \]

Now, we write the incompressibility condition as
\[ \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_z = 0 \]

or
\[ \dot{\varepsilon}_{xx} - \left[ \frac{\sigma_x}{E} - \frac{E}{E} (\dot{\sigma}_y + \dot{\sigma}_z) \right] + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_z = 0 \]

or
\[ \dot{\varepsilon}_{xx} - \left( \frac{\sigma_x}{E} - \frac{E}{E} (\dot{\sigma}_y + \dot{\sigma}_z) \right) + \frac{1}{n} \left[ \left( \sigma_{kl} n_{kl} \right)_{yy} + \left( \sigma_{kl} n_{kl} \right)_{zz} \right] = 0 \]
\[
\begin{align*}
\dot{\mathbf{e}}_{kl} & = \sigma'_{kl} \\
& = \sigma_{11} n_{11} + \sigma_{22} n_{22} + \sigma_{33} n_{33} + \sigma_{13} n_{13} + \sigma_{23} n_{23} + \sigma_{31} n_{31} \\
& = \sigma_{11} n_{11} + \sigma_{22} n_{22} + \sigma_{33} n_{33} + 2 \sigma_{13} n_{13} \\
\varepsilon_{ij} & = \left< \sigma_{kl} \mathbf{m}_{kl} \right> n_{ij} \\
\varepsilon_{yy} & \rightarrow \varepsilon_{33} = ? \\
\left< \sigma_{kl} \mathbf{m}_{kl} \mathbf{m}_{ij} \right> = \\
& = \left[ \sigma_{11} n_{11} + \sigma_{22} n_{22} + \sigma_{33} n_{33} + 2 \sigma_{13} n_{13} \right] n_{22} \\
& \quad + \left[ \sigma_{11} n_{11} + \sigma_{22} n_{22} + \sigma_{33} n_{33} + 2 \sigma_{13} n_{13} \right] n_{33} \\
& = \sigma_{11} n_{11} (-n_{11}) + \sigma_{22} n_{22} (-n_{11}) \\
& \quad + \sigma_{33} n_{33} (-n_{11}) + 2 \sigma_{13} n_{13} (-n_{11}) \\
\end{align*}
\]
Substituting, we obtain

\[ E_x - \left( \frac{\sigma_x}{E} - \mu \left( \frac{\sigma_y + \sigma_z}{E} \right) \right) \]

\[ + \frac{1}{h} \left[ -\sigma_{xx} n_{xx}^2 + \sigma_{yy} n_{yy} n_{xx} \right. \]

\[ - \sigma_{zz} n_{zz} n_{xx} - 2 \sigma_{xy} n_{x} n_{xx} \right] = 0 \]

Rearranging, we obtain

\[ E \Delta E_x + \Delta \sigma_{xy} \left[ n_{yy} n_{xx} E + \mu \right] \]

\[ + \Delta \sigma_{zz} \left[ n_{zz} n_{xx} E + \mu \right] - \Delta \sigma_x \left[ \frac{n_{xx}^2 E}{h} + 1 \right] \]

\[ = \frac{2}{h} \Delta \sigma_{zz} n_{xx} n_{x} \]

or

\[ E \Delta E_x - \frac{2}{h} E \Delta \sigma_{zz} n_{xx} n_{x} + \Delta \sigma_{zz} \left[ \mu + \frac{n_{zz} n_{xx} E}{h} \right] \]

\[ = \Delta \sigma_x \left[ 1 + \mu \frac{n_{xx} E}{h} \right] + \Delta \sigma_y \left[ \frac{n_{xx} n_{xx} E}{h} - \mu \right] \]

This is 2nd eqn 13 in Jiang et al's paper. Note the typo in the paper.
So our equations will be

using \( \dot{\varepsilon}_y = 0 \) (plane strain condition)

\[
\dot{\varepsilon}_y^e + \dot{\varepsilon}_y^p = 0
\]

\[
\frac{\sigma_y}{E} - \frac{\mu}{E} \left( \ddot{\sigma}_x + \ddot{\sigma}_z \right) + \frac{1}{h} \left( \ddot{\sigma}_{kl} n_{kl} \right) n_{z2} = 0
\]

or

\[
\frac{\sigma_y}{E} - \frac{\mu}{E} \left( \ddot{\sigma}_x + \ddot{\sigma}_z \right) + \frac{1}{h} \left( \ddot{\sigma}_{11} n_{11} + \ddot{\sigma}_{22} n_{22} + \ddot{\sigma}_{33} n_{33} + 2 \ddot{\sigma}_{13} n_{13} \right) n_{z2} = 0
\]

\[
\Delta \frac{\sigma_y}{E} \left[ \frac{1}{E} + \frac{n_{33}^2}{h} \right] + \Delta \sigma_z \left[ -\frac{\mu}{E} + \frac{n_{33} n_{z2}}{h} \right] + \Delta \sigma_x \left[ -\frac{\mu}{E} + \frac{n_{11} n_{z2}}{h} \right] + \Delta \varepsilon_{x3} \left[ 2 n_{13} n_{z2} \right] = 0
\]
Rearranging,

\[ \Delta \sigma_y \left( 1 + \frac{E_{nyy}}{\mu} \right) + \Delta \sigma_x \left[ \frac{n_{xx} \sigma_y}{\mu} - \mu \right] \]

\[ = \Delta \sigma \left[ \mu - \frac{n_{33} n_{yy} E}{\mu} \right] \]

\[ - \Delta T_{xy} \left[ 2 n_{x3} n_{yy} E \right] \]

This is 2nd eq. of equation (13).

(note the typo in the paper).

Same equation is also given as equation (5) in the paper without a typo.
After the elastic loading analysis, we find that $E x b$, $\sigma_z b$, $E x z b$ is non-zero. So we get from (1) to (2).

The equations are two unknowns, but we need to enforce this at different depth $3/a = 0.1$ to 2, so there are more than two equations. For each depth we have two equations and two unknowns.

Then, we evaluate

$$\Delta \varepsilon_z = f(\Delta \sigma_z, \Delta \sigma_y),$$

$$\Delta \lambda_{xz} = f(\Delta \sigma_x, \Delta \sigma_y).$$
Few comments on the MT and JS formulations:

(1) MT formulations, using the PR equations, is for isotropic hardening.

(2) In MT paper the PR equations are written in a different form than the standard form we presented in class.

\[ T_2 = \frac{1}{2} (S_{x}^2 + S_{y}^2 + S_{z}^2) + T_{x_3}^2 = k^2 \]

(note our coordinate frame is that of JS, x rolling direction, z is depth direction)

(3) Few typos in the paper:

Starting with \( dE_i^P = dl S_{ij} \)

Define \( W = S_{ij} dE_i^P \)

\[ 2G W = 2G S_{ij} S_{ij} dl \]

\[ W = 2k^2 dl \]

\[ dA = W / 2k^2 \]

(There is a typo in Eq 5.9 Prager.)

Then, looking at Mermin Johnson (plastically incompressible case)

\[ E_i^j = E_i^j - E_i^p \]
\[ S_x = 2G \left[ \varepsilon_{xx} - \frac{W}{2k^2} S_x \right] \]

The sign is + in MJ paper which is a typo.

(4) Why are the residual stresses compressive and what is their role?

As residual builds up, \( \sigma_3 - (6x + 6x_r) \) decreases. If the shear stress decreases, higher pressure is needed for yielding.
AN ANALYSIS OF PLASTIC DEFORMATION IN ROLLING CONTACT

By J. E. Merwin, M.S., Ph.D.* and K. L. Johnson, M.Sc.Tech., Ph.D., M.A. (Associate Member)†

When two metal cylinders roll together under a contact pressure sufficient to cause yielding, a surprising mode of plastic deformation occurs. The surface of each cylinder is progressively displaced in the forward direction of rotation relative to the core by plastic shearing in a thin subsurface layer. This phenomenon was first observed by Crook (6) in 1957 and the results of a more complete experimental investigation are reported by Hamilton (9) in an accompanying paper. In this paper an attempt is made to explain the mechanism of 'forward flow' by an approximate numerical analysis of the elastic–plastic stress cycles to which the material is subject in repeated rolling contact. On the basis of an idealized material which is elastic–perfectly plastic and isotropic, it is shown that a forward displacement of the surface would be expected as a result of the complex cycle of stress and strain encountered in rolling. It is also shown that residual compressive stresses are introduced into the immediately subsurface layers during the first few cycles of the load. If the load does not exceed the elastic limiting load by more than 66 per cent further plastic deformation would then cease. At higher loads a steady pattern of plastic deformation is predicted which in its principal features is consistent with the observed behaviour.

INTRODUCTION

If two metal cylinders are rolled together under a sufficiently large pressure to cause yielding of the material, an interesting type of deformation takes place. This deformation is principally one in which an elastic surface layer rotates with respect to the elastic core of the cylinder. The two elastic portions are separated by a narrow band of plastically deformed material. In the absence of any tangential forces, the direction of rotation of the surface relative to the core is in the direction of rotation of the cylinder.

These observations, when they were first described by Crook (6)‡ and Welsh (7), gave rise to considerable speculation as to why a purely normal force acting between the cylinders should produce an asymmetrical pattern of deformation and, further, why the flow should occur in the forward direction. For example, it was suggested that a lubricant film between the discs would lead to an asymmetrical distribution of contact pressure, but experiments reported by Welsh showed that 'forward flow' was observed in the absence of a lubricant. Again, the Bauschinger effect and also the upper-yield-point phenomenon were cited as possible causes of unsymmetrical behaviour in a loading cycle in which the shear stress reverses its direction.

However, as stated at the time by Johnson (8), it is reasonable to expect an asymmetrical contact stress field on general grounds. The occurrence of repeated plastic deformation, whether cumulative or not, involves continuous energy dissipation. Thus, for steady rolling motion to proceed external work must be supplied, which demands that the reaction force between the discs be displaced from the line of centres towards the entry side. A further consequence of plastic deformation is that the state of stress at any point in the field depends upon the entire loading history of an element of material before it reaches the point in question. It appeared to us, therefore, that any detailed and convincing

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‡ References are given in the Appendix.
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picture of the process must trace the stress and strain history of material elements as they pass repeatedly through the contact stress field.

In the meantime Hamilton* (9) has conducted an extensive experimental investigation of the phenomenon which provides a sound basis for analysing the mechanism of the process. Hamilton has shown that the phenomena occurs with all of a variety of metals and is not restricted to steel discs used by Crook and Welsh. He observed that the deformation in the first cycle of load was markedly different from subsequent loadings. A steady state was then quickly reached when the total forward displacement of the surface layer became cumulative in the sense that it was directly proportional to the number of revolutions of the cylinder. If two discs of the same metal were geared together so that they ran at exactly the same speed their surfaces became corrugated, a subsidiary effect which interfered with the process of forward flow. It was found that the development of corrugations could be eliminated either by permitting a slight amount of 'slip' between the discs or by rolling against a harder disc which did not deform plastically. When the tendency to corrugate was prevented, the forward displacement per cycle was found to be independent of the speed of rolling. Finally, by measuring the rolling friction, Hamilton estimated the energy dissipated in plastic deformation. This was found to be appreciably greater than the work which would be necessary to produce the observed forward displacement by a process of simple shear.

Hamilton's experiments appear to support the hypothesis that, in the absence of corrugations, forward flow arises from the elastic-plastic stress-strain cycle in rolling contact rather than from 'special' material properties. It was felt, therefore, that a relatively simple idealization of an elastic-plastic material could be employed in an analytical study of the process.

Before proceeding to the analysis, certain other features of the process may be anticipated in qualitative terms. At small loads the contact stresses will be everywhere elastic. With increasing load some point in the stress field will reach its yield point, which may be determined from the elastic stresses and a suitable yield criterion. This load will be referred to as the elastic limit. At higher loads some plastic flow will occur, thereby leaving residual stresses in the surface region of the cylinders after the load has passed. During subsequent passes of the load the material is subject to the combination of residual and contact stresses which together may not reach the yield point. In other words the system 'shakes down' to a resultant state of stress which is entirely elastic. The maximum load at which shakedown occurs will be termed the shakedown limit. Clearly, loads at which cumulative plastic flow occurs must exceed the shakedown limit.

It is well known that in elastic contact of solids the yield point is first reached beneath the surface. Hence, for loads not greatly in excess of the elastic limit, plastic deformation will be fully 'contained' by elastic material and the order of magnitude of elastic and plastic strains will be the same. Up to now solutions in the mathematical theory of plasticity of problems of contained plastic deformation have been obtained only for some simple cases. An exact solution, in which conditions of equilibrium, compatibility and elastic-plastic constitutive relations are simultaneously satisfied throughout the solid and on the boundaries, seems to be out of the question at the present time. A principal difficulty arises from the unknown shape and location of the elastic-plastic interface. However, an approximate solution has been found which exhibits all the important features of the observed behaviour and, for loads not greatly exceeding the shakedown limit, is not likely to differ appreciably from an exact solution.

The problem of two rolling cylinders will be simplified by the following idealization:

1. The system is replaced by a rigid cylinder rolling on the plane surface \( y = 0 \) of a semi-infinite solid \( y < 0 \) (see Fig. 21 for the co-ordinate system and the notation for stresses).
2. The material of the solid is elastic-perfectly plastic (i.e. non-work-hardening) and isotropic.
3. The deformation is plane, hence the axial strain \( \varepsilon_z \) is zero and all other stresses and strains are independent of \( z \).

The implications of these idealizations will be discussed later.

**ELASTIC CONTACT**

The elastic solution for the contact of two circular cylinders or a cylinder with a plane is well established and is derived from the classical Hertz theory. Since the contact region is small compared with the other dimensions, a cylinder can be considered as a semi-infinite solid which has a prescribed normal displacement on the surface of the form \( w = \alpha - \beta x^2 \) over the interval \( |x| < a \) where \( \alpha \) and \( \beta \) are constants and \( a \) is the semi-contact width. Hertz obtained the pressure distribution in the contact region necessary to produce the

![Fig. 21. Rigid cylinder rolling on an elastic–perfectly plastic plane](image-url)
prescribed displacements. Later Beliaev (10), Radzimovsky (11) and others obtained solutions for the complete internal stress field by various methods. These solutions are exact for a rigid cylinder in contact with a semi-infinite solid and closely applicable to two elastic cylinders in the vicinity of the contact zone.

In plane strain the elastic stresses may be expressed:

\[
\begin{align*}
\sigma_x + \sigma_y &= \frac{4P}{\pi a} \mathcal{R} \left( e^{-i\zeta} \right) \\
\sigma_y - \sigma_x + 2\tau_{xy} &= \frac{2P}{\pi a} i \cosh \zeta - \cosh \bar{\zeta} e^{-i\zeta}
\end{align*}
\]  

(1)

where \( x + iy = a \cosh \zeta, \zeta = \xi + i\eta \) and \( P \) is the contact force per unit axial length of the cylinder.

In this paper, the value of Poisson's ratio, \( \nu \), is taken to be 0.3.

We can now determine the elastic limit. The yield condition chosen is that proposed by von Mises which states that the second invariant of the stress deviation tensor, \( J_2 \), cannot exceed some limiting value \( k^2 \). The second invariant is given by

\[
J_2 = \frac{1}{2} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 2\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 \right]
\]

(2)

In simple shear \( J_2 = \tau^2 \), whilst in simple tension \( J_2 = \frac{1}{2} \nu^2 \).

Hence \( k \) may be interpreted as either the yield stress in simple shear or the yield stress in tension divided by \( \sqrt{3} \).

In Fig. 22 are shown contours of constant values of \( \sqrt{J_2} \) expressed in terms of the maximum contact pressure \( p_0 \). It can be seen in the figure that yielding will first occur on the line of centres at a depth \( y = 0.705a \) when

\[
p_0 = 3.10k
\]

In other words, only direct stresses acting parallel to the surface can remain, but they may vary, independently of each other, with depth below the surface. Similarly, the possible residual strains are:

\[
\begin{align*}
(\varepsilon_y)_r &= f_6(y), \quad (\varepsilon_{y\theta})_r = f_4(y), \\
(\varepsilon_x)_r &= (\varepsilon_z)_r = (\varepsilon_{x\theta})_r = (\varepsilon_{z\theta})_r = 0
\end{align*}
\]

(5)

The only non-zero residual strains are the direct strain \( (\varepsilon_y)_r \) which accounts for compression of the solid normal to its surface, and the shear strain \( (\varepsilon_{y\theta})_r \) which, if it occurred with each repeated passage of the load, would produce a cumulative tangential displacement of the surface.

An insight into the way in which the residual stresses arise can be obtained from an inspection of the elastic stress field. To simplify the discussion, let us temporarily consider only stresses in the \( x-y \) plane.

The principal shear stress in the \( x-y \) plane is given by

\[
\tau = \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}
\]

(6)

Its maximum value lies on the \( y \)-axis where \( \tau_{xy} = 0 \). The variation of \( \tau \) along the line of centres is shown in Fig. 23. The absolute maximum value occurs at a depth \( 0.786 \) and is equal to \( 0.300 p_0 \). It should be noted

† This simplification amounts to substituting the Tresca maximum shear stress criterion of yield, \( \tau_{\text{max}} = k \), for the von Mises criterion.
that this maximum shearing stress, which occurs on the line of centres, is not the shear associated with the cumulative strain \( \gamma_{yu} \) observed in Crook's experiments. This stress, which occurs on planes at 45° to the surface, leads to a compressive displacement normal to the surface. Clearly the normal displacement must be rather small and reach a limiting value, since the dimensions of either a cylinder or a plane solid cannot decrease indefinitely. Eventually, therefore, residual stresses must exist which eliminate further yielding in the normal direction.

If we indicate some yield stress \( k \) in Fig. 23 it will be seen that \( \tau \) exceeds the yield criterion in the layer lying between \( y_1 \) and \( y_2 \). Now the addition of residual compressive stresses

\[
(\sigma_z)_r = 2(k - \tau) \quad \ldots \ldots \quad (7)
\]

would eliminate any further yielding on the \( y \)-axis. However, a residual stress which is capable of eliminating further yielding due to stresses on the \( y \)-axis, will not necessarily prevent yielding on either side where \( \tau_{yu} \neq 0 \). To illustrate this point we examine the variation of elastic stress with \( x \) at a constant depth \( y = -a \) in Fig. 24. Initially the principal shear stress \( \tau \) exceeds \( k \) by a maximum amount at \( x = 0 \). The addition of a residual stress \( (\sigma_z)_r \), given in equation (7) reduces the value of \( \tau \) to \( k \) on the \( y \)-axis, but it still exceeds \( k \) on either side, in the regions where \( \tau_{yu} \) has its maximum values. It follows from equation (6) that no possible residual stress can prevent yield if \( (\tau_{yu})_{\text{max}} \) exceeds \( k \), but for \( (\tau_{yu})_{\text{max}} \ll k \), a residual stress can be found which will reduce the resultant value of \( \tau \) to less than \( k \) at all points along \( y = -a \).

The same reasoning applies to other depths, from which it follows that a residual stress distribution can be found which will eliminate yield according to the Tresca criterion provided that the maximum value of \( \tau_{yu} \) anywhere in the field is less than \( k \). Conversely, if \( (\tau_{yu})_{\text{max}} \) exceeds \( k \), then no residual stress system can prevent subsequent yielding.

Fig. 23. Variation of elastic stresses with depth along the axis of symmetry

Principal shear stress \( \tau \) is modified to \( \tau' \) by the addition of the residual stress \( (\sigma_z)_r \), taken from Fig. 23.

Fig. 24. Variation of elastic stresses at a constant depth

\[ y = -a \]
relations will be employed. It is well to recall the postulates in the plastic zone, the Prandtl-Reuss incremental is the von Mises yield criterion and associated flow rule, and let boundary displacements within the contact region which introduce the stress deviation and strain deviation. If we stress deviations are e, boundary Free from traction. These boundary conditions calls for comment; it may be interpreted as the rate at which the stresses do work in connection with a change of shape. During plastic deformation the von Mises flow condition ensures that the elastic strain energy of distortion remains constant, therefore \( \dot{W} \) corresponds to the rate of irreversible energy dissipation through plastic action.

Turning now to the rolling problem, the stress and strain rates may be expressed:

\[
\frac{d}{dt} [\epsilon_{12}, \ldots \gamma_{25}; \epsilon_{13}, \ldots \gamma_{25}] = U \frac{\partial}{\partial x} [\epsilon_{13}, \ldots \gamma_{25}; \epsilon_{13}, \ldots \gamma_{25}]
\]

where \( U \) is the steady peripheral velocity of rolling. Substitution in equations (13) replaces the time rates of change by stress and strain gradients with respect to \( x \). Hence, time, do not then appear in the stress–strain relations. Since the strains are assumed to be known (equal to the elastic strains) throughout the field, if the state of stress is known at any point in the plastic zone, the rate of change of stress deviation with \( x \) at that point is given by equations (11). A step-by-step numerical analysis may now be performed for the stress cycle of an element at any depth \( y \). Starting from the elastic stress state of the element at the point \( x_0 \) when it first reaches the yield point (\( J_3 = k^2 \)) on the entry side, the stress rates of change with \( x \) are found from equations (11) and used to predict the values of the stress components when the element has moved a small distance \( \Delta x \). Thus, step by step, the stress variations with \( x \) for constant \( y \) are obtained, as illustrated in Fig. 25a–d.

At this point we might consider the implications of assuming that the strains are the same as in the purely elastic problem. Obviously these strains constitute a compatible system; consistent and physically reasonable stress–strain relations have been used; the boundary conditions (prescribed surface displacements within the contact area, zero surface tractions outside) have been satisfied. The solution is inexact, therefore, only to the extent to which the stresses do not satisfy conditions of equilibrium. Clearly no attempt is made to maintain equilibrium during the strain cycle, since stresses at one level are being modified independently of stresses at other levels. However, it is possible to restore equilibrium at the end of each loading cycle.

If the loading cycle had been entirely elastic the stresses would approach zero after the element had passed under the load. As a result of the plastic deformation, however, residual stresses \( (\sigma_x)' \), \( (\sigma_y)' \), \( (\sigma_z)' \), and \( (\tau_{xy})' \) are found as \( x \to \pm \infty \) in the above-described computation. Such stresses do not satisfy the conditions of equilibrium of a plane surface free from traction. These boundary conditions

\[
W' = s_s \dot{\varepsilon}_s + s_y \dot{\varepsilon}_y + s_z \dot{\varepsilon}_z + \tau_{xy} \dot{\gamma}_{xy} . \tag{12}
\]

and a dot over any quantity denotes a time rate of change of that quantity. These relations apply during plastic deformation, in so long as \( J_3 = k^2 \) and \( \dot{W} > 0 \). For \( J_3 < k^2 \) or \( \dot{W} < 0 \) the elastic equations (9) apply. The quantity \( \dot{W} \) calls for comment; it may be interpreted as the rate at which the stresses do work in connection with a change of shape. During plastic deformation the von Mises flow condition ensures that the elastic strain energy of distortion remains constant, therefore \( \dot{W} \) corresponds to the rate of irreversible energy dissipation through plastic action.

For the application of these equations it is convenient to introduce the stress deviation and strain deviation. If we let \( s = (\sigma_x + \sigma_y + \sigma_z)/3 \) and \( e = (\epsilon_x + \epsilon_y + \epsilon_z)/3 \), then the stress deviations are \( s_x = \sigma_x - s \), etc. and the strain deviations are \( e_x = \epsilon_x - e \), etc. For plane strain the usual Hooke’s law stress–strain relation for elastic deformation may be written:

\[
\begin{align*}
\tau_{xy} &= G \gamma_{xy}, \\
\sigma_x &= 2Ge_x, etc.
\end{align*}
\]

and

\[
\dot{\gamma}_{xy} = G \left[ \dot{\gamma}_{xy} + \frac{\dot{W}}{k^2} \tau_{xy} \right]
\]

where

\[
\dot{W} = s_s \dot{\varepsilon}_s + s_y \dot{\varepsilon}_y + s_z \dot{\varepsilon}_z + \tau_{xy} \dot{\gamma}_{xy} . \tag{12}
\]

and a dot over any quantity denotes a time rate of change of that quantity. These relations apply during plastic deformation, in so long as \( J_3 = k^2 \) and \( \dot{W} > 0 \). For \( J_3 < k^2 \) or \( \dot{W} < 0 \) the elastic equations (9) apply. The quantity \( \dot{W} \) calls for comment; it may be interpreted as the rate at which the stresses do work in connection with a change of shape. During plastic deformation the von Mises flow condition ensures that the elastic strain energy of distortion remains constant, therefore \( \dot{W} \) corresponds to the rate of irreversible energy dissipation through plastic action.

Turning now to the rolling problem, the stress and strain rates may be expressed:

\[
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\]

where \( U \) is the steady peripheral velocity of rolling. Substitution in equations (13) replaces the time rates of change by stress and strain gradients with respect to \( x \). Hence, time, do not then appear in the stress–strain relations. Since the strains are assumed to be known (equal to the elastic strains) throughout the field, if the state of stress is known at any point in the plastic zone, the rate of change of stress deviation with \( x \) at that point is given by equations (11). A step-by-step numerical analysis may now be performed for the stress cycle of an element at any depth \( y \). Starting from the elastic stress state of the element at the point \( x_0 \) when it first reaches the yield point (\( J_3 = k^2 \)) on the entry side, the stress rates of change with \( x \) are found from equations (11) and used to predict the values of the stress components when the element has moved a small distance \( \Delta x \). Thus, step by step, the stress variations with \( x \) for constant \( y \) are obtained, as illustrated in Fig. 25a–d.

At this point we might consider the implications of assuming that the strains are the same as in the purely elastic problem. Obviously these strains constitute a compatible system; consistent and physically reasonable stress–strain relations have been used; the boundary conditions (prescribed surface displacements within the contact area, zero surface tractions outside) have been satisfied. The solution is inexact, therefore, only to the extent to which the stresses do not satisfy conditions of equilibrium. Clearly no attempt is made to maintain equilibrium during the strain cycle, since stresses at one level are being modified independently of stresses at other levels. However, it is possible to restore equilibrium at the end of each loading cycle.

If the loading cycle had been entirely elastic the stresses would approach zero after the element had passed under the load. As a result of the plastic deformation, however, residual stresses \( (\sigma_x)' \), \( (\sigma_y)' \), \( (\sigma_z)' \), and \( (\tau_{xy})' \) are found as \( x \to \pm \infty \) in the above-described computation. Such stresses do not satisfy the conditions of equilibrium of a plane surface free from traction. These boundary conditions

\[
W' = s_s \dot{\varepsilon}_s + s_y \dot{\varepsilon}_y + s_z \dot{\varepsilon}_z + \tau_{xy} \dot{\gamma}_{xy} . \tag{12}
\]
require \((\sigma_y)_r = (\tau_{xy})_r = 0\) (see equations (4)). The computation gives non-zero values \((\sigma_y)_r\) and \((\tau_{xy})_r\), because elastic recovery of strains has been assumed at the outset. In order to satisfy the condition of equilibrium governing the residual stresses, the strains are permitted to relax elastically until \((\sigma_y)_r = (\tau_{xy})_r = 0\). In this relaxation process plane strain demands \((\epsilon_y)_r = 0\) and symmetry conditions demand that \((\epsilon_y)_r = 0\) (see equation (5)). The residual strains \((\epsilon_y)_r\) and \((\gamma_{xy})_r\) are thus given by:

\[
\begin{align*}
(\epsilon_y)_r &= -\frac{(1-2\nu)}{2(1-\nu)G} (\sigma_y)_r, \\
(\gamma_{xy})_r &= -\frac{(\tau_{xy})_r G}{G}
\end{align*}
\]

Furthermore, the residual stresses \((\sigma_z)_r\) and \((\sigma_x)_r\), as initial conditions in a repeated numerical integration, the result of a second passage of the load can be calculated giving new values of residual stresses. This procedure is repeated until there is no further change in \((\sigma_x)_r\) and \((\sigma_z)_r\), i.e. the increment of \((\epsilon_y)_r\) per pass approaches zero. However, the increment of \((\gamma_{xy})_r\) per pass, the residual shear strain which produces a tangential displacement of the surface, does not approach zero but reaches some constant value. Therefore, the overall shear displacement is cumulative with repeated traversals of load. The direction of \((\gamma_{xy})_r\) corresponds to a 'forward' displacement as observed by Crook. If the tangential displacement of the surface per pass is denoted by \(\delta\), it may be found from the steady-state values of \((\gamma_{xy})_r\) by integration through the depth of plastic deformation, namely:

\[
\delta = \int_{y_1}^{y_2} (\gamma_{xy})_r \, dy . \quad \ldots \quad (16)
\]

Curves of \(\sigma_y\), \(\sigma_x\), \(\sigma_z\), and \(\tau_{xy}\) at a depth of \(y = -a\) are shown in Fig. 25 for several traversals of the load. A steady state is approached rapidly.

An unfortunate feature of the method, inherent in its basic approximations, is that the theoretical pressure distribution over the surface of contact cannot be determined. However, the variation of \(\sigma_y\) on a representative plane, \(y = \) constant, in the plastic zone (e.g. \(y = -a\), shown in Fig. 25b) is suggestive. For equilibrium, the integral of \(\sigma_y\) from \(x = -\infty\) to \(x = +\infty\) must balance the normal contact force. In the steady state, at least, it appears that the magnitude of the force is not very different from the equivalent elastic case. It is reassuring to note that the centre of pressure has been shifted towards the inlet side which overall energy considerations demand.

An energy balance may be used, in the steady state, to calculate the moment which must be applied to the cylinder to overcome the resistance to rolling due to plastic deformation. If this moment per unit axial length is denoted by \(M\),
recalling that the rate of energy dissipation per unit volume is given by $\dot{W}$ in equation (12), external work may be equated to energy dissipated:

$$M \mu/R = \int \dot{W} \, dx \, dy \quad (17)$$

where the integration is taken throughout the plastic zone. From equation (13) $\dot{W}$ is seen to be proportional to $U$, so that $M$ is, in fact, independent of rolling speed. The quantity $M/R$ may be interpreted as the force of rolling resistance.

**RESULTS OF THE COMPUTATIONS AND DISCUSSION**

The integration of repeated stress cycles was programmed for the Cambridge University 'Edsac' computer. The Runge-Kutta-Gill technique was used in the integration process with an interval $\Delta x = 0.01a$. The plastic zone was covered at depth intervals of $y = 0.20a$. The computation at each depth was repeated until an approximately steady state was reached (usually about four passes) giving rise to stress cycles of the form illustrated in Fig. 25. The build-up of residual stresses $(\sigma_3)_r$ and $(\sigma_2)_r$, with repeated loading is shown in Fig. 26. The whole computation was performed for three values of applied load factor, namely

$$p_0/k = 4.00, 4.76 \text{ and } 5.55$$

In this expression $p_0$ denotes the value of the maximum Hertzian contact pressure assuming elastic behaviour. The actual maximum pressure may differ slightly from this value.

For the steady-state condition, the forward tangential displacement of the surface per pass given by equation (16) and the rolling resistance, given by equation (17), are plotted against the load factor in Figs 27 and 28 respectively.

The characteristic features of the steady-state pattern of plastic deformation are illustrated in Fig. 29 for a load factor

---

**Fig. 26. Build-up of residual stress $(\sigma_3)_r$ and $(\sigma_2)_r$ with repeated passages of the load**

---

(a) $p_0 = 4.0k$ (shakedown limit); (b) $p_0 = 4.8k$; (c) $p_0 = 5.5k$. 

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Fig. 27. Forward flow per cycle in the steady state computed at the values of \( p_0/k \) shown.

Fig. 28. Rolling resistance due to plastic energy dissipation computed from the elastic-plastic stress cycles, at the values of \( p_0/k \) shown.

\( p_0/k = 5.56 \). The layer in which cumulative plastic action takes place is thinner than the layer which has experienced initial plastic flow. Referring to Fig. 25d, the plastic component of shear strain \( \gamma_{yz} \) is proportional to the difference between the actual and the corresponding 'elastic' stress. The steady-state variation of \( \gamma_{yz} \) and also \( \gamma_{yy} \) for the deformation cycle at \( y = -a \) are shown in Fig. 29b. It will be seen that direction of plastic shearing reverses. The residual forward flow arises from the fact that the amount of forward shear outweighs that of backward shear. However, the plastic work done in the reversal of the direction of shearing, together with the plastic work done by the other stress components, ensures that the rate of total plastic work greatly exceeds that which would be required to produce the forward displacement in simple, uni-directional shear. The ratio of work to produce forward flow to the total work is given by:

\[
\frac{k\delta R}{M} = \left( \frac{8G}{p_0} \right) \left( \frac{M G}{R \sigma_0^b} \right) \left( \frac{p_0}{k} \right)
\]

The authors have frequently been asked whether they can give a simple explanation of the forward displacement without recourse to the complexities of the complete analysis. With this object in view the following greatly simplified model is offered. In the steady state the only component of strain which can lead to cumulative deformation is the shear \( \gamma_{yz} \). We shall now follow the loading history of a typical material element in the plastically deformed layer and ignore the existence of all other stress and strain components. This reduces the cycle from one of complex stress to one of simple shear. If, as the complete analysis assumes, the strain cycle is symmetrical the element will be subject to equal and opposite peak values of shear strain \( \gamma_{yz} \). The stress-strain cycle is shown in Fig. 30. OA represents elastic loading at entry up to the yield stress \(-\sigma_0\), AB is backward plastic shear, BC is elastic stress reversal to the yield stress \(+\sigma_0\) and CD is forward plastic shear. Since no residual \( \tau_{yz} \) stress can exist, elastic unloading at exit brings

...
The element acquires a permanent strain OE. The $\tau_{xy} - \gamma_{xy}$ cycle taken from the computations for $p_0 = 5.56k$ and $y = -a$ is plotted to the same scale for comparison (broken line).

**Fig. 30. Simple shear cycle OABCDE in which a material element of yield strength k is subject to a reversal of strain between limits $\pm \gamma_{xy}$ and finally left in a stress-free state (E)**

the element to the state at point E in which it has acquired a forward (+ve) residual strain OE.

For comparison the $\tau_{xy} - \gamma_{xy}$ cycle at $y = -a$ taken from the complete analysis is plotted to the same scale in Fig. 30. Although the simple shear cycle underestimates the residual forward shear strain by about one half, it exhibits the same qualitative features; a reversal of strain in which the forward shear outweighs the initial backward shear. It is immediately evident that the model predicts a value of $\frac{a}{2}$ for the ratio of forward flow work to total work. It may be recollected that in the complete analysis the strain cycle is continued to the point $F'$ where $\gamma_{xy} = 0$. The residual stress $(\sigma_{xy})'$ is subsequently relaxed (equation (14)) to return to state $E'$.

When comparing the quantitative results of the computations with experimental observations, the limitations of the theory should be kept in mind*. The assumption of elastic strains which leads to stresses which do not satisfy equilibrium has already been emphasized. It is difficult to estimate the error thereby incurred, although Hamilton’s photoelastic experiments suggest that the actual strain distribution is not perfectly symmetrical. It is likely, however, that the lack of symmetry and the violation of equilibrium in the steady state will be appreciably less than during the first passage of the load. The probable influence of asymmetrical strain can be inferred from the simple model of Fig. 30. Thus an increase in the peak value of $\gamma_{xy}$ on the entry side and a corresponding decrease on the exit side would reduce the forward displacement but increase the plastic work input.

The analysis has been carried out for a semi-infinite solid with a plane surface. For the contact of two cylinders the curved profile affects the residual stress system given in equation (4), since the presence of a circumferential stress $(\sigma_\theta)$ necessarily introduces a radial stress $(\sigma_r)$. However, provided that the depth of the plastic layer is small compared with the radius of the cylinder, i.e. within the restrictions of the Hertz theory, the radial stress will be negligibly small.

With two cylinders of identical materials, so that yield occurs simultaneously in each, the boundary displacement within the contact region will conform to a circular profile as assumed in the analysis. If one cylinder yields before the other the geometry of the interface will be distorted, but it is unlikely that the effect on the strain field will be appreciable.

The conditions of plane deformation assumed in the analysis are not likely to be approached near to the ends of contacting cylinders. In consequence of the high hydrostatic component of stress a fair bulk of material is required to provide lateral support and to prevent sideways plastic flow.

The remaining limitations of the analysis are restricted to the material properties. The experimental evidence that a variety of metals, whether hardened or not, exhibit the phenomenon of forward flow suggests that the influence of hardening properties is not of primary importance. The step-by-step nature of the calculation would enable a law of work hardening to be incorporated without fundamental difficulties, but, in view of the present uncertainty in specifying a hardening rule which is physically reasonable in a cycle where the shear stresses reverse, the labour involved would hardly be justified. To apply the results to a particular metal which work hardens, the question of an appropriate value for $k$ arises. Initial yield will obviously be governed by the primitive yield point of the material, but both shakedown and cumulative forward flow represent steady-state conditions after repeated loading when considerable plastic straining will have taken place. A value of $k$ associated with the metal in a strain-hardened state would then be more appropriate. For example, the yield stress deduced from a Vickers hardness indentation by the method due to Tabor (16) might provide a reasonable value to use.

**CONCLUSION**

The investigation set out to provide a rational explanation of the puzzling phenomenon of cumulative plastic deformation which is observed when metal cylinders are in rolling contact. The analysis which has been presented is not exact in the sense demanded by the mathematical theory of plasticity, but nevertheless follows an accepted practice for the approximate solution of problems in applied mechanics by starting from an assumed pattern of deformation which is both kinematically compatible and intuitively reasonable. Its success must be measured by the fact that it predicts qualitatively, and to some extent quantitatively, the principal features of the phenomenon; the build-up of residual stresses in a layer below the surface, cumulative displacement of the surface in the forward direction of motion and resistance...
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to rolling through plastic dissipation in a well-defined subsurface layer.

Since the analysis has been based upon the stress–strain relations of an ideal, isotropic, elastic–perfectly plastic material, it may be concluded that the phenomenon of forward flow is predominantly due to the complex cycle of stress to which the material is subject in rolling contact. Special properties of the material such as strain hardening or softening, the Bauschinger effect, or an upper yield point, which have at various times been invoked to explain the unsymmetrical shear deformation, whilst possibly contributing to the details of the behaviour, are not necessary for its occurrence and are unlikely to play a dominant role.

The deformation processes which, according to the analysis, take place when two cylinders roll freely together under a load not greatly in excess of the elastic limit (certainly for \( p_0 < 6k \)) may now be described.

During the first passage of the rolling contact load the material deforms plastically in a layer whose depth below the surface roughly corresponds to that in which the Hertzian elastic stresses, could they be realized, would exceed the yield limit of the material. During this deformation the surface is compressed only slightly below its original level and residual elastic compressive stresses acting parallel to the surface are introduced. This form of deformation continues, to a decreasing extent, with the next four or five cycles of load, after which a steady state is reached when there is no further change in residual stress or surface depression. If the load is within the shakedown limit (\( p_0 < 4k \)) no further plastic flow takes place. Subsequent deformation is then entirely elastic.

At higher loads a steady pattern of plastic action is repeated with each passage of the load. This is restricted to a layer which is thinner and centred closer to the surface than initially. The significant component of plastic strain is now one of shear parallel to the surface. On entering the plastic zone each material element is sheared backwards in the sense which the elastic shear stress at entry suggests. However, before the axis of symmetry is reached the direction of the plastic shear straining reverses. Throughout the exit side the material is subject to a severe shearing in the forward direction, which more than eliminates the backward shear obtained at entry and gives rise to the residual plastic shear strain which accumulates with repeated passages of the load to produce a forward displacement of the surface.

The material is subject to a cycle of reversed shear whose range is appreciably greater than the residual strain increment at the end of the cycle. The energy dissipated during this cycle of plastic action is three to four times as great as would be required to produce the forward displacement in simple unidirectional shear. This energy is responsible for the resistance to rolling; it is manifest by an offset in the centre of contact pressure towards the entry side.

ACKNOWLEDGEMENTS

We wish to acknowledge the help given by Mr G. M. Hamilton, Dr W. Hirst and Dr A. W. Crook of the A.E.I. Research Laboratory, Aldermaston. The results of Hamilton's experiments, reported in his paper (9), were placed at our disposal as they became available and many valuable discussions were held.

We are especially indebted to Mrs G. M. Birch for her work in programming the numerical computations for the Edsac 2 computer.

APPENDIX

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Communications

Professor G. Chabert (Paris)—These papers indicate that, in the case of gear wheels, rolling should cause plastic flow of the underlayer towards the base of the teeth of the driving element, and towards the top of the teeth of the receiving element. However, according to Crook (6), there is a superimposed slip effect which causes creep, tending to move the underlayer away from the pitch circle of the driving element and to concentrate it towards the pitch circle of the receiving element. This effect is, of course, well known and has been observed for many years with gear wheels.

In any event, the two effects are cumulative at the bottom of the tooth spaces of the two elements, and opposed at the top of their teeth.

This observation, in fact, led Crook to state: 'The type of deformation found with the rolling discs is of interest since it suggests a new explanation of the greater propensity to pitting of discs with the lower peripheral speed and of the dedenda of gear teeth.'

However, the mathematical analysis by Dr Merwin and Dr Johnson (second paper) produces figures showing that this argument cannot be maintained. The phenomenon of continuous creep, demonstrated for the first time with rolling discs by Mr Hamilton's firm and discussed in both papers, should be observable only with such heavy loading as to be met exclusively in industrial applications where case hardening is necessary.

To show this, let $R$ be the ultimate tensile stress and $\sigma$ the yield stress (approximately $0.85R$ for low-carbon alloyed steels designed for heavy-loaded applications). Continuous creep takes place when

$$P_0 = 4k = \frac{4\sigma}{\sqrt{3}} = \frac{4 \times 0.85R}{\sqrt{3}} = \frac{3.4R}{\sqrt{3}}$$

Then

$$\left(\frac{P_0}{R}\right)^2 = \frac{(3.4)^2}{3} = 3.85$$

Now let us take the load criterion introduced in S.E.I.E. Bulletin No. 35, p. 8:

$$\Delta = \frac{1}{2\pi} \left(\frac{P_0}{R}\right)^2 = \frac{3.85}{2\pi} = 0.62$$

This value corresponds to heavy loading applications such as heavy commercial vehicles or tractors. Case hardening is of general practice for such gears.

Mr G. J. Moyar (Schenectady, New York)—In their analysis of a complex problem, Dr Merwin and Dr Johnson have demonstrated considerable ingenuity in the selection of pertinent approximations which still maintain many essential features of the actual phenomenon.

With regard to the method of determining stresses in the elastic–perfectly plastic material, have the authors considered the possible simplification offered by a piece-wise linear yield surface and associated flow rule? With such theories (17) it is possible to express the stress components directly in terms of the strains and thus eliminate the step-by-step machine computation. It would be necessary to follow the stress point from one regime (face or intersections of the yield polyhedron) to another during the strain cycle. These theories, for example the Tresca yield and associated flow rule, are physically acceptable according to Drucker's stability postulate. In fact, considering the complex loading cycle, there is little experimental evidence to justify the selection of the Prandtl–Reuss theory over such theories.

I should like to amplify some of the discussion of material hardening behaviour. Although many metals are not closely described by perfectly plastic behaviour, they may attain a saturation hardness, as the authors suggest, after which the analysis is appropriate. Since there is some indication that steady-state cyclic strain accumulation is not possible with many hardening rules, such a suggestion has practical merit. To illustrate the behaviour when an idealized hardening, as opposed to a hardened material, is treated, consider the result of isotropic (18) and kinematic (19) hardening on the basis of the simple model proposed by the authors. Three cycles for each case are illustrated in Fig. 31. The isotropic model rapidly 'shakes down' to purely elastic behaviour. In the case of kinematic hardening a stable cycle of energy dissipation is established although forward strain accumulation ceases.
Despite the shortcomings of the current hardening idealizations it does not necessarily follow that the inherent hardening properties are not significant. That steady-state strain accumulation is possible, even without macroscopic stress or strain gradients, has been discussed previously (20) and explored experimentally (21). Yet this qualification does not detract from the authors’ analysis of the forward shear phenomenon. Their theory is direct and convincing without appeal to mysterious material properties.

REFERENCES


Mr E. Ollerton, B.Sc. (Associate Member)—Mr Hamilton is to be congratulated on the clarity of his photoelastic fringe patterns. The production of clear fringe patterns in this type of problem can be very difficult. The disc can warp in its own plane under the radial load, giving a contact area, which is trapezoidal instead of rectangular. If this occurs the fringe pattern becomes indistinct because the stresses vary through the disc thickness. Mr Hamilton evidently avoided this difficulty.

In the calculation of a pressure distribution which would cause the observed tilting of the fringe pattern, it has been assumed that the principal directions of stress in the contact surface are radial and tangential; in other words, that no shear stresses were transmitted across the contact surface. The validity of this assumption can be tested very easily by viewing the disc in plane-polarized light, with the polarizer and analyser aligned in the radial and tangential directions. The resulting isoclinic lines will connect all points at which the principal directions of stress are radial and tangential, and if an isoclinic extends across the full width of the contact surface, the assumption is justified.

In the absence of such a justification, the inferred pressure distribution should be treated with some caution,
because the tilting of the fringe pattern could well be due to the combined effect of pressure redistribution and surface shear stresses. The application of a driving torque to the disc, tending to roll it to the right in Fig. 19 would itself cause the fringe pattern to tilt in the observed direction. Could Mr Hamilton say whether the loading rig was capable of transmitting such a torque?

Fig. 32 shows a fringe pattern obtained by myself by loading an Araldite disc against a mild-steel plane and applying a tangential force. The disc did not roll or slide and the loading conditions did not produce any plastic deformation in the steel. The applied tangential force was equivalent to a torque tending to roll the Araldite disc to the left, so the fringe pattern should be inverted for comparison with Fig. 19. The ratio between tangential and radial force was about 0.1, and the coefficient of friction about 0.3. At almost limiting tangential forces the fringe patterns agreed closely with those predicted by Poritsky (22).

Mr N. A. Scarlett (Chester)—Having read the two papers I would congratulate the authors on the original and interesting work described.

My only comment is that similar corrugations to those described have been experienced on hypoid gears run in the laboratory of my firm’s research centre under slow-speed heavily loaded conditions.

Dr D. Tabor (Cambridge)—I should like to express my great pleasure on reading these two papers on rolling-contact phenomena in the plastic range. They make a very real and much-needed contribution to our understanding of the processes involved when one body rolls on another. I hope Mr Hamilton will not think me unappreciative of his experimental skill if I devote my remarks to the paper by Dr Merwin and Dr Johnson. They have tackled a difficult and puzzling experimental observation and have provided an explanation that is lucid and convincing. Further they have kept very clearly before the reader the nature and the limitations of the assumptions they have made.

When Dr Eldredge and I began our study of rolling friction about 10 years ago we studied in particular the rolling of a hard steel sphere over the flat surface of a softer metal. There were three clear-cut experimental observations. First the initial traversal generally produced a permanent groove in the lower surface. Secondly the rolling friction for this traversal was not reduced by lubricants. These led us to the view that the major part of the rolling resistance was not due to interfacial slip but to plastic displacement of metal ahead of the ball. There was, indeed, good quantitative agreement between the observed friction and that calculated on this assumption.

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The third observation was that with repeated traversals of the same track the track width generally increased and the rolling friction diminished until after about 100 traversals a steady state seemed to have been reached. At this stage it appeared as if the contact were determined by elastic deformation of the ball and the groove. The elliptic region of contact agreed well with that given by the Hertzian relation. Again the rolling friction was little affected by lubricants. It was natural, therefore, to look upon this as an extension of the plastic case and to consider the rolling friction as arising essentially from elastic deformation and the hysteresis losses associated with it. This explained all the major observed results. The main defect, however, was that in some cases the hysteresis losses were enormous: indeed, with copper at higher loads one would need to assume that practically the whole of the elastic energy of deformation was lost during rolling.

In order to study the elastic hysteresis mechanism in greater detail we left metals and concentrated our study on rubber. It soon became clear that the hysteresis explanation for rubber-like materials was descriptively valid and finally that it was quantitatively valid. Here, however, a new point emerged. Every element passing under the rolling member is subjected to a reversal of the shear direction and because of this the hysteresis losses are about three times higher than simpler theories would suggest. These results also led to the view that the relatively high rolling frictions previously observed with metals in the near-equilibrium condition were probably due to reversed plastic shear of the elements under the rolling member. However, no quantitative calculations of this were attempted.

In the paper by Dr Merwin and Dr Johnson we have, for the first time, such a quantitative estimate carried out with great perception and ingenuity. It is interesting to see how far their results may indeed explain the relatively high frictions we observed. If we approximate our narrow ellipse of contact to a parallel band of contact we may use their results of Fig. 28. Assuming at this stage that $\rho_0/h$ was about 5, the observed friction for copper is of the same order as that shown in Fig. 28.

This raises two interesting additional points. As I understand it the energy dissipated by plastic deformation is provided by strain energy in the elastic hinterland. This energy loss is equivalent to throwing the centre of pressure ahead of the geometric centre of contact. There must be a limit to this. Clearly the centre of pressure cannot be beyond the outer edge of the region of contact and even this is physically unacceptable. A more reasonable limit would be when the centre of pressure is about $\pi/2$ ahead of the geometric centre of contact. This would give a resistive couple equal to $Pa/2$. An analysis we made in our earlier paper suggests a smaller upper limit of $3p/16a$. In terms of Fig. 28 the rolling resistance parameter $MG/Rp_0^2$ in this case has a value of about 0.16. This implies that one could not easily dissipate more energy than this without so greatly modifying the elastic strains that the rest of the analysis would be of dubious validity.
Fortunately this is above the calculated values given by Dr Merwin and Dr Johnson but not very much (Fig. 33). Secondly, of course, the analysis does not say anything about energy losses below the elastic or shakedown limit. In this range the rolling friction is due to hysteresis losses and as a matter of interest I have drawn the loss to be expected for an effective hysteresis loss of 2 per cent. It is seen how small this is compared with the value obtained by Dr Merwin and Dr Johnson in the plastic range.

I should also like to congratulate the authors on the elegant explanation they give for cumulative plastic shear in the surface layers even though elastic equilibrium has, otherwise, been achieved. As they point out, this contributes a fair part to the total friction. Even more important is the part it must play in the fatigue of the surface. Here I am thinking not so much of fatigue in rolling friction itself, but rather of fatigue and wear of sliding surfaces in the presence of effective lubrication. For even if the interfacial adhesion is trivial the continuous passage of one set of asperities over another will produce stress patterns resembling those which occur in rolling friction. It is possible to regard the normal loading-unloading cycle of the surface asperities as the cause of fatigue and wear and this has been done in an interesting way by Rozeanu (23). But if surface shearing occurs in the way described by Dr Merwin and Dr Johnson it could be a far more effective cause.

Finally I should like to add two questions. Do the authors consider it possible to carry out a similar treatment for a hard sphere rolling over a flat surface? Do they think a shakedown condition is theoretically possible? Our own early measurements suggested a near-equilibrium state after about 100 traversals but later work showed a slow but continuous diminution in friction and growth of track width for another quarter of a million traversals.

Secondly, the results of Figs 27 and 28 suggest that if rolling involves initial plastic deformation there is a great advantage in using a material which has a large work-hardening index so that in approaching the shakedown condition the pressure becomes less than \( 4k \) where \( k \) is the augmented critical shear stress for the work-hardened material. Do the authors know of any practical experience (say from ball-bearing studies) which would support this?

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**Authors' Replies**

**Mr G. M. Hamilton**—Professor Chabert has raised the important question of the loads at which disc machine tests should be conducted. This is a contentious matter but it is well known that to reproduce failures such as pitting and scuffing the loads applied between the discs have to be higher than the nominal values for the gears which are being simulated. The normal interpretation of this fact is that the load on the gear in the region where the failure occurred is considerably greater than average, generally presumed to be due to misalignment or inaccurate manufacture. In these circumstances the entire load can be supported on as little as 10 per cent of the available face width. Benson (24) has reported an example of a gear failure with subsurface deformation, very similar to the type described in the paper, which shows that such loads are realized in practice. From this point of view it would be interesting to know whether the corrugations on the hypoid gear mentioned by Mr Scarlett were accompanied by subsurface deformation.

Many of the questions raised by Mr Ollerton can be settled by reference to Fig. 34 showing a sketch of the loading rig used for the photoelastic tests which was designed especially to reduce the shear stresses transmitted across the contact. The specimen was driven under load across the surface of the glass disc which in turn was free to roll across the surface of a hard steel plate, so that its rolling resistance was negligible. With this arrangement such rolling resistance as remains causes the fringes to tilt.
in the opposite direction to that shown in Fig. 19 and in fact stress patterns very similar to the one shown by Mr Ollerton were obtained by restraining the glass disc in its passage across the steel plate.

Dr J. E. Merwin and Dr K. L. Johnson—We would certainly agree with Professor Chabert that frictional tractions due to sliding play an important part in the process of plastic deformation of rolling surfaces. The present work has been extended to include sliding friction and is reported by Johnson and Jefferis (25). In that investigation it was shown that the value of $p_0$ at the shakedown limit is decreased from $4k$ by the action of a friction force, falling to $3k$ and $1.8k$ for coefficients of sliding friction of 0.25 and 0.5 respectively.

Dr Moyar's suggestion of using the Tresca criterion and flow rule has been found useful in investigating surface yielding produced by friction forces. Here the expressions for the elastic strains are simple. Beneath the surface numerical analysis seems unavoidable and the Prandtl-Reuss equations, being without discontinuities, are particularly convenient for numerical integration. We are grateful for Dr Moyar's neat extension of our simple model of 'forward flow' to examine the effect of strain hardening. The 'kinematic hardening' model represents, in a very idealized way, the Bauschinger effect displayed by most metals undergoing reversed plastic strain. Although Dr Moyar is correct is stating that the model as shown predicts that forward flow (but not plastic deformation) would eventually cease, in order to reach this limiting state, it would be necessary for the metal to harden to at least twice its original yield stress. It might be mentioned that two mild steel discs which were tested showed a different behaviour from copper and duralumin discs. Instead of the forward flow being directly proportional to the number of stress cycles (see Fig. 10), it showed a marked decrease after a large number of cycles. Presumably this effect is due to the hardening characteristics of mild steel.

The authors are grateful to Dr Tabor for his communication relating his pioneering work on rolling friction to the present theory. The agreement with Dr Tabor's rolling resistance measurements for copper is encouraging since the comparison with Mr Hamilton's measurements in Fig. 17 is rather poor. Very reasonably Dr Tabor uses the asymmetry of the pressure distribution to set an upper limit for the validity of the theory at a load corresponding to $p_0/k \approx 6$. Further support for this figure is provided by the fact that at this load a subsurface residual stress equal to the yield stress would be induced, so that at higher loads the pattern of deformation would be likely to show a marked change.

To perform a similar analysis for the three-dimensional problem of a sphere rolling over a flat surface would seem to be a formidable undertaking at the present time. The residual stress pattern introduced by the formation of a groove will be complex and difficult to analyse. However, the problem is under consideration and some preliminary observations have been made in reference (25).

The idea of using a material which is initially soft, deforms to produce a conforming groove, and thereby hardens to prevent further deformation, is an interesting one. But it does not seem to fit in with ball-bearing experience. Here the fatigue life improves with increasing initial hardness. Also the reversed shear beneath the surface has been observed to soften the material in that region (see Bush, Grube and Robinson (26)).

REFERENCES

An Analytical Approach to Elastic-Plastic Stress Analysis of Rolling Contact

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Introduction

In rolling line contact, when the contact load exceeds the shakedown limit, incremental surface displacements will accrue with each rolling pass, concurrent with cyclic plastic strains and residual stresses. The rolling contact stresses and the repeated plastic deformation of the material govern the initiation of contact fatigue. We therefore seek simple yet practical methods of determining residual stress/strain and ratcatching per cycle.

Merwin and Johnson (1963) were the first to develop an approximate method for the determination of plastic deformation in rolling contact by assuming that the strain cycle remains identical with the elastic strain cycle. A relaxation procedure followed to meet the boundary conditions of the residual stresses and strains upon passage of load. With this method, Johnson et al. (Merwin and Johnson, 1963; Johnson and Jefferis, 1963) achieved results which were confirmed by the recent finite element simulations (Bhargava et al., 1983). The predicted cumulative surface displacements using this method were reported in agreement with the experimental results (Hamilton, 1963). Since the Merwin-Johnson method predicted a surface displacement rate smaller than the finite element predictions, Hearle and Johnson (1987) forwarded an alternate method to estimate the subsurface plastic shear strain rate in rolling line contact by assuming that the orthogonal shear strain component is the only plastic strain occurring throughout the cycle. Following Hearle and Johnson (1987), Bower and Johnson (1989), realizing the difference between subsurface and surface plastic flow, introduced two different methods to deal with them separately. They applied Hearle and Johnson’s concept on subsurface flow but included strain and cyclic hardening by employing a nonlinear kinematic hardening rule described by Bower (1987, 1989). For the surface plastic flow, they assumed that within the thin layer just beneath the contact surface, the stresses conform to the elastic solution. With this assumption, the stress histories on the thin layer of contact were determined (Bower, 1987), yet with a lack of specificity on “surface,” “subsurface,” and “near surface” (Bower and Johnson, 1989). In addition, stress and deformation should be continuous at any time; however, using two different methods to deal with surface and subsurface does not guarantee such smoothness.

Using a nonlinear kinematic hardening model, McDowell and Moyar (1986, 1991) developed a new algorithm to determine the residual stresses and strains in rolling line contact. Referring to the coordinate system presented in Fig. 1, they assumed that under elastic-plastic contact loading the two stress components \(\sigma_x\), the stress in radial direction, and \(\tau_{xz}\), the orthogonal shear stress, are identical to the elastic solutions. In addition, the strain rates in the rolling (\(x\)) and axial (\(y\)) directions are maintained zero. These conditions allow all the stress components to be determined from the elastic solutions. We note that enforcing \(\epsilon_x = 0\) is not necessary for obtaining
highly history dependent and nonproportional loadings such as observed in rolling contact. Despite the advances in constitutive modeling, for the rolling contact problems the commercially available finite element software has utilized relatively simple plasticity theories. Attempts to employ refined constitutive models in FEM results in extremely long computation times (Hahn, 1992) especially for residual stresses and ratcheting over many cycles. Therefore, further developments in semi-analytical methods for rolling contact analysis continue to be appealing.

In summary, the determinations of stress-strain histories remain a fundamental problem in fatigue analysis of rolling contact. Currently, the agreements between the analytical solutions and those of finite element analyses are not satisfactory. On the other hand, the finite element method applied to contact stress analysis has the shortcomings of long computational times. Development of an approximate analytical method for elastic-plastic stress analysis of contact is inviting, and this is the subject of this paper.

The Analytical Approach

The new approach for elastic-plastic stress analysis of rolling contact consists of a stress invariant assumption and a stress and strain relaxation procedure. First, we propose that the stress cycle during elastic-plastic rolling contact is equal to the elastic solution. Expressing this assumption mathematically,

$$\sigma_{ij} = \sigma'_{ij}$$

where $\sigma_{ij}$ represent the components of the stress tensor and $\sigma'$ denotes elastic solution. We, then, proceed with a relaxation procedure where the stresses and strains at a point are relaxed proportionally to meet the residual stress and strain conditions upon each passage of load.

The rolling line contact is taken to illustrate the new approach. The coordinate system on a semi-infinite half plane is shown in Fig. 1(a). The rolling body translates in the negative x-direction. The normal load is assumed to be elliptically distributed over a contact length of $2a$ and translates in the positive x-direction. The maximum Hertzian pressure is represented by $p_0$. The total normal load is $P$ and the total tangential force is denoted by $Q$. Initially, all stress and strain components are set to zero. The stress components at point A are determined using formula based on elasticity theory such as those by Smith and Liu (1953). We assume no stick/slip and ignore non-Hertzian contact effects. When $x$ is larger than about 10a the stress state will be elastic. As the load translates rightwards and point A nears $x = 0$ plastic deformation will occur. When the load moves further to the right, and $x$ reaches about $-10a$, the stress state will become elastic again. Generally, at this moment, based on the elastic calculation (Eq. (1)), the stress and strain components are non zero. However, because of the geometrical conditions, some residual stresses and strains

**Nomenclature**

- $a$ = half width of contact area in line contact
- $E$ = elasticity modulus
- $G$ = elasticity modulus in shear
- $H$ = plastic modulus
- $k$ = yield stress in pure shear
- $n_{ij}$ = normal on the yield surface
- $P$ = total normal load in line contact
- $p_0$ = maximum Hertzian pressure in line contact
- $Q$ = total tangential force in line contact
- $R$ = yield surface size in two surface model
- $R^*$ = radius of bounding surface in two surface model
- $S_{ij}$ = deviotoric stress components
- $\sigma_{ij}$ = center of yield surface in deviotoric space
- $\delta$ = surface displacement rate
- $\Delta(\gamma_{xz})$ = residual shear strain rate in line rolling contact
- $\sigma_r, \sigma_\tau, \tau_{xz}$ = orthogonal stress components
- $\sigma_{ij}, r, (\sigma'_r), r, (\sigma'_r), r, (\sigma'_r), r$ = residual stress components
- $\epsilon_1, \epsilon_2, \gamma_{xz}$ = orthogonal strain components
- $\epsilon_r, (\epsilon'_r), (\epsilon'_r)$ = residual strain components
- $\mu$ = Poisson’s ratio
should be zeros. For example, in line rolling contact the residual strain in the rolling (x) direction and residual stress in the radial (z) direction should be zero. Therefore a procedure is adopted to enforce the residual stress and strain conditions through proportional relaxation. The outcome of the relaxation procedure is the "correct" residual stresses and strains for that pass. The same routine is repeated for the second and subsequent passages of the load with the previous residual stresses assigned to be the initial values for the next passage of the load. This procedure is continued until the residual stresses and strains reach a stabilized state. Note that the approach enforces equilibrium at the end of each load pass. The compatibility of the strains is not strictly satisfied in the present model.

We note in Fig. 1 (b) that a driving wheel is subjected to a tangential force opposite to its rolling direction, i.e., Q/P > 0, and a driven wheel undergoes a tangential force coincident with the rolling direction, i.e., Q/P < 0.

When applying the proposed method to the plane stress rolling line contact, Eq. (1) becomes,

\[ \sigma_x = \sigma_x^e, \quad \sigma_y = \sigma_y^e, \quad \tau_{xz} = \tau_{xz}^e \]  

where \( \sigma_x^e, \sigma_y^e, \text{ and } \tau_{xz}^e \) are elastic solutions. The plane strain condition requires,

\[ \dot{\varepsilon}_y = 0 \]  

where a dot above a symbol means derivative with respect to time. In the elastic regime the above equation will result in,

\[ \dot{\varepsilon}_y = \mu (\dot{\varepsilon}_x + \dot{\varepsilon}_z) \]  

where \( \mu \) is the Poisson’s ratio. When the deformation is elastic-plastic, the stress increment \( \dot{\sigma}_y \) can be expressed as,

\[ \dot{\sigma}_y = \left( \frac{1}{E} \right) \dot{\varepsilon}_x + \left( \frac{1}{E} \right) \dot{\varepsilon}_z - 2 \frac{E}{h} n_{xz} \tau_{xz} + \frac{E}{h} n_y^2 \tau_{xz} \]  

where \( E \) is the elasticity modulus, \( h \) is the plastic modulus function, and \( n_{xz} \) represent the components of the unit normal in the plastic strain rate direction on the yield surface. It should be noted that in deriving Eq.(5) we identified the plastic strain rates as normal to the von Mises yield surface. The normality flow rule can be written,

\[ \dot{\varepsilon}_y = \frac{1}{h} \left( \dot{S}_{ij} n_{ij} \right) n_{ij} \]  

where \( \dot{\varepsilon}_y \) is the MacAuley bracket, that is, \( \langle \phi \rangle = 0.5 (\phi + |\phi|) \). \( \dot{S}_{ij} \) in Eq. (6) are the components of the deviatoric stress which are defined as,

\[ \dot{S}_{ij} = \dot{\sigma}_{ij} - \frac{1}{3} \dot{\varepsilon} \sigma_{kk} \]  

where \( \sigma_{ij} \) are the components of the deviatoric back stress and \( k \) is the yield stress in pure shear. The von Mises yield surface is defined as,

\[ f = \frac{1}{2} (S_{ij} - \alpha_{ij}) (S_{ij} - \alpha_{ij}) - k^2 = 0 \]  

and isotropic hardening is ignored.

When the results from different approaches are compared, it is advisable to apply identical material properties and plasticity theories. In the finite element analyses (Bhargava et al., 1990; Hahn et al., 1987; Kumar et al., 1989; Ham et al., 1989), exercised as the reference for comparisons, the linear kinematic hardening rule was used,

\[ \dot{\alpha}_{ij} = \left( \dot{S}_{ij} n_{ij} \right) n_{ij} \]  

According to Merwin and Johnson (1963), the residual stresses and strains in line rolling contact should satisfy the following conditions,

\[ (\sigma_x)_0 = 0; \quad (\sigma_y)_0 = f_1(z); \quad (\sigma_z)_0 = f_1(z); \quad (\tau_{xz})_0 = f_2(z); \quad (\tau_{yz})_0 = f_3(z) \]  

where \( f(z) \) implies that \( f \) is a function of \( z \) only, and the subscript \( r \) outside the bracket represents residual stress or strain.

When Eq. (1) is invoked, the residual stress and residual strain components at a point are generally non-zero, and do not satisfy Eq. (11). At the end of the rolling pass, we denote the non zero stress and strain components as \( (\sigma_x)_b, (\sigma_y)_b, (\sigma_z)_b, (\tau_{xz})_b, (\tau_{yz})_b \), and set \( M \) steps from the beginning of the relaxation to its end. The following stress and strain increments are enforced until Eq. (11) is satisfied,

\[ \Delta \sigma_x = \frac{(\sigma_x)_b}{M}; \quad \Delta \sigma_y = \frac{(\sigma_y)_b}{M}; \quad \Delta \tau_{xz} = \frac{(\tau_{xz})_b}{M} \]  

where \( \Delta \) denotes finite increment. Corresponding to the above four increments, the increments of \( \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \text{ and } \tau_{xz} \) can be computed. When the deformation is elastic, Hooke’s law applies to determine the stress increments, \( \Delta \sigma_x, \Delta \sigma_y, \text{ and } \Delta \tau_{xz} \). For elastic-plastic deformation, \( \Delta \sigma_x \) and \( \Delta \sigma_y \) is obtained by solving the following equation family,

\[ \Delta \sigma_x = \frac{E}{h} (\sigma_{xx})_{b} + \frac{1}{E} \left( \frac{E n_{xx}}{h} \right) (\partial n_{xx}) = \frac{E}{h} (\sigma_{xx})_{b} \]  

where \( n_{xx} = n_{11}, n_{yy} = n_{22}, n_{zz} = n_{33}, \text{ and } n_{xz} = n_{13} \) and are determined from Eq. (8). The \( \Delta \sigma_x \) and \( \Delta \tau_{xz} \) are obtained as follows,

\[ \Delta \sigma_x = \frac{1}{E} (\sigma_{xx})_{b} + \frac{1}{E} (\sigma_{xx})_{b} + \frac{1}{E} (\sigma_{xx})_{b} + \frac{1}{E} (\sigma_{xx})_{b} \]  

and

\[ \Delta \tau_{xz} = \frac{2 n_{xx}}{h} \]  

where the strain increments are available from Eqs. (12) and (13).

Comparison of Analytical Results With FEM and Experiments

Hahn et al. (Bhargava et al., 1990; Hahn et al., 1987; Kumar et al., 1989; Ham et al., 1989) conducted a number of finite element elastic-plastic stress analyses of rolling line contact problems. Their results are reproduced here and compared with those obtained from the Merwin-Johnson method, the McDowell-Moyar method, and the proposed method. The comparisons are summarized in Figs. 2 through 4.
Because the FEM analyses (Bhargava et al., 1990; Hahn et al., 1987; Kumar et al., 1989; Ham et al., 1989) support linear kinematic hardening rule, the same stress relation and hardening rules are applied in the calculations using the Merwin-Johnson method, the McDowell-Moyar method, and the proposed method. As in the FEM analyses, the tangential force is assumed to be proportional to the normal load in the analytical calculations. The residual stresses after the 11th passage of the load are perceived to be the steady state for both the Merwin-Johnson and the proposed methods. The residual stress state after the 21st passage of the load is chosen to be the steady state when the McDowell-Moyar method is exercised. With the McDowell-Moyar method, more passages of the load are usually necessary to reach a steady state. Normalized quantities are employed throughout the paper. Stresses and Hertzian pressure are normalized with respect to the yield stress in shear, \( k \), and plastic modulus is normalized with respect to the elasticity modulus in shear, \( G \). All lengths are normalized with respect to \( a \), the half width of the contact area.

\( M \) in Eq. (12), the number of steps in the relaxation procedure of the proposed method, was set to be 200 for all the cases investigated. It has been checked that when \( M \) is larger than a certain value, the results would not be altered. For the rolling contact cases investigated, \( M > 50 \) is sufficient.

In Fig. 2, results from Kumar et al. (1989) FEM analyses are reproduced and compared with those obtained by using different analytical methods. Both the residual stresses in the rolling direction, \( (\sigma_z)_{z} \), and the axial direction, \( (\sigma_y)_{z} \), are demonstrated. The material is a bearing steel which has a yield strength in pure shear, \( k \), of 606 MPa and the plastic modulus, \( H \), of 25.7 G, where \( G \) is the elasticity modulus in shear. The tangential force, \( Q \), is negative according to the coordinate system selected in Fig. 1. In the case illustrated in Fig. 2, \( Q/P = -0.2 \) and \( \rho_0/k = 4.5 \), where \( \rho_0 \) is the maximum Hertzian pressure. From Fig. 2, the Merwin-Johnson method predicts practically the same residual stress zone size as the FEM but underestimates greatly the residual stress in the rolling \( (x) \) direction. The location of maximum residual stress in the rolling \( (x) \) direction is about 0.4a beneath the contact surface according to the Merwin-Johnson method while the FEM predicts the value to be about 1.0a. Based on the McDowell-Moyar method, the stress in rolling \( (x) \) direction is always identical to the stress in the axial \( (y) \) direction. This prediction by the McDowell-Moyar method clearly deviates from the FEM results. Also, the FEM predicts an enhanced plastic zone size compared to the McDowell-Moyar method.

The residual stresses predicted by the proposed method conform to the FEM predictions. Residual stress results in both the rolling \( (x) \) and the axial \( (y) \) directions are comparable to the FEM data, and furthermore, the agreements on locations of the maximum residual stress and the size of the residual stress zone are impressive.

Figure 3 summarizes the steady state residual stresses predicted by the FEM, the Merwin-Johnson method, and the proposed method for the pure rolling \( (Q/P = 0) \) contact. The FEM results are from Hahn et al. (1987). The material properties are the same as that affiliated with Fig. 2. A larger load, \( \rho_0/k = 6.0 \), was applied. In Fig. 4, the FEM results are from
Ham et al. (1989) on carbon rail steel. The yield stress in shear, \( k \), is 139 MPa and the plastic modulus, \( H_p \), is 1.64 G. We consider the case \( p_0/k = 4.5 \) and \( Q/P = 0 \). In both cases shown in Figs. 3 and 4, the Merwin-Johnson method predicts lower residual stress in both the rolling \((x)\) direction axial \((y)\) directions compared to FEM. Residual stress zone sizes are in close agreement. The proposed method and the FEM predictions of residual stresses in both the rolling \((x)\) and the axial \((y)\) directions once again bear striking resemblance. More comparison have been made (Jiang and Sehitoglu, 1992a), which lead to the same conclusion.

The FEM results in Figs. 2 through 4 point out that the residual stress magnitudes are highly dependent on the plastic modulus; with the same load ratio \( p_0/k \), the larger the plastic modulus, the smaller the residual stresses. The residual stress in the rolling \((x)\) direction is different from the residual stress in the axial \((y)\) direction in both the magnitude and distribution pattern. According to the McDowell-Moyar method (1986 and 1991) the stress in the rolling \((x)\) direction is always equal to the stress in the axial \((y)\) direction. Comparing with the FEM, Merwin-Johnson method can capture the axial residual stress better than the residual stress in the rolling direction. In addition, the predicted sizes of the residual stress zones by the Merwin-Johnson method are in close accord with the FEM and the proposed method. However, the peak residual stress in the rolling direction calculated by the Merwin-Johnson method are only about 0.25 to 0.40 of the values of the FEM predictions. In the rolling direction, the Merwin-Johnson method finds the locations of the peak residual stress to be around 0.2 to 0.6a beneath the contact surface, while both the FEM and the proposed method establishes those locations to be about 0.9a ~ 1.2a beneath the surface.

Johnson et al. (Johnson and Jeffris, 1963; Pomeroy and Johnson, 1969), Hamilton (1963) and Shima et al. (1981) all compared their theoretically calculated results obtained from the Merwin-Johnson method using an elastic-perfectly plastic material model with experimental data on a strain hardening material. Clearly, their comparisons can be viewed more qualitatively than quantitatively, considering that the material description employed in the analysis influence the computed residual stresses. It is therefore imperative that different approaches and experiments employ consistent material properties.

If the FEM results are designated as the reference for comparisons, the proposed method removes many of the limitations of the McDowell-Moyar and the Merwin-Johnson approaches. To check the ability of the proposed approach further, we present in Figs. 5 and 6 the experimental residual stresses from Pomeroy and Johnson (1969) along with the analytical predictions by the proposed method. Experimental measurements of the circumferential and axial components of residual stress were reported in a disk under sliding rolling contact. Two different materials were studied in the experiments; one was aluminum-magnesium alloy HE20WP and the other is 0.3 percent carbon steel EN 5A in the normalized and stress-relieved condition. Because the material constants for
the elastic-plastic stress analysis were not available in the reference (Pomeroy and Johnson, 1969), in the theoretical analysis using the proposed method the stress-strain relation of the material is assumed to be bilinear. The plastic modulus of the aluminum alloy is assumed to be 2.0G. The plastic modulus of the carbon steel is assumed to be 3.0G. A two surface plasticity model to be described later is employed in the analysis. In the analysis the tangential force is assumed to be proportional to the normal load. The experimental stresses and the calculated quantities of the carbon steel are contrasted in Fig. 5. According to Pomeroy and Johnson (1969), \( p_0/k = 5.5 \) and \( Q/P = -0.1 \). The experimental results in Fig. 5 confirm that the axial residual stress is larger than the circumferential residual stress. Favorable comparisons of experimental stresses and the model predictions of aluminum alloy in Fig. 6 substantiate the validity of the model.

For an approximate method, it is inevitable that some conditions of equilibrium and/or compatibility are violated. For
example, the Merwin-Johnson method does not enforce the equilibrium conditions and the proposed method does not guarantee the compatibility conditions during the loading cycle. Due to the complicated stress and strain histories involved in the rolling contact, it is difficult to evaluate a method based on how much the basic condition (equilibrium and compatibility) is violated. Therefore, the theoretical validity of the assumption and the relaxation procedure, on which the proposed approach is based, will not be discussed. More concerns have been placed on the agreement between the results obtained from the approach and those obtained from finite element analyses. However, several characteristics of the proposed method should be pointed out. The fact that both the Merwin-Johnson and the proposed method can predict stress zone sizes practically identical to those predicted by the FEM indicates that beyond the plastic zone in the rolling line contact, the stresses can be accurately described by the formula of Smith and Liu (1953) as if there were no existence of the plastic zone. Because the plastic zone is small and the stresses in this small zone should be continuous, it then becomes reasonable that the stresses in the plastic zone should not be quite different from the elastic solutions. The close agreements of the results of the proposed method and those from the FEM as shown in Figs. 2 through 4 and with those from experiments as shown in Fig. 5 and Fig. 6 provide strong positive support for this assumption. The proposed method, due to the elastic stress assumption, can only be applied to the cases where the materials strain harden. This condition limits little on its applications because there is in fact no engineering material that behaves exactly elastic-perfectly plastic, especially considering materials under cyclic loading. Moreover, the proposed theory is limited to small plastic deformation problem, because only under small plastic deformation could the stresses in the small plastic zone be approximated by elastic solutions. The small plastic deformation condition depends on the normal load, tangential traction, and material properties, and is difficult to define quantitatively with a simple formula.

The Merwin-Johnson method assumes elastic strain, therefore the compatibility conditions are satisfied (kinematically admissible). The proposed method, on the other hand, assumes elastic strain and accordingly the equilibrium conditions are satisfied (statically admissible). Analogous to the principle in plasticity of upper and lower bounds given by solutions that are either kinematically admissible or statically admissible, the Merwin-Johnson method may provide lower bounds and the proposed method the upper bounds. We can find some evidence to this effect in Figs. 2-4. However, this may not be a general conclusion and we have found it difficult to theoretically confirm such upper and lower bounds assertion because (i) both the Merwin-Johnson and the proposed methods enforce the boundary conditions by a stress/strain relaxation procedure which makes the resultant residual stresses and strains both kinematically and statically admissible, and, (ii) the deformation in rolling contact is path dependent and the stress-strain relations are nonlinear.

Two Surface Nonlinear Kinematic Hardening Model

Motivated by the encouraging outcome of the proposed method, a detailed calculation of the residual stresses, and surface displacements is undertaken with a more realistic constitutive model. We employ a two surface plasticity model, comparable to McDowell and Moyar model (1986 and 1991), describing the material behavior of 1070 steel (Jiang and Sehitoglu, 1992a, Sehitoglu and Jiang, 1992).

A two surface plasticity model is shown in Fig. 7. The terms \( n, S, \alpha \) in Fig. 7 represent second order tensors. The increment of deviatoric back stress in the two surface model is expressed according to Mroz hardening rule (Mroz, 1967), i.e.,

\[
\alpha_{ij} = \frac{S_{ij}n_{ij}}{\nu_{mn}P_{mn}} n_{ij}
\]

where

\[
\nu_{ij} = S_{ij} - S_{ij}
\]

is the direction of translation of the inner surface;

\[
S_{ij} = \frac{R_{max}}{R} (S_{ij} - \alpha_{ij})
\]
where $S_{ij}^*$ is a point on the bounding surface $f^*$. This surface is defined as,

$$f^* = \frac{3}{2} S_{ij}^* S_{ij} - R_{\text{max}}^2 = 0$$

where $R_{\text{max}}$ is the bounding surface radius given by,

$$R_{\text{max}} = \max \left( \frac{3}{2} S_{ij} S_{ij} \right) \text{ if } \max \left( \frac{3}{2} S_{ij} S_{ij} \right) > R^*$$

For the sake of simplicity, the plastic modulus function suggested by Drucker and Palgen (1981) is used,

$$h = \frac{2}{3} H = \frac{W}{(\sqrt{J_2}/k)^b}$$

For 1070 steel, typically used in railroad wheels, we use the following cyclic stress-strain parameters,

$$G = 80 \text{ GPa}; \mu = 0.3; k = 139 \text{ MPa}; \frac{H}{G} = \frac{4.11}{(\sqrt{J_2}/k)^{2.1}}; R^* = 4R$$
The residual stress and strain state saturates after the 11th passage of the load and represents the steady state. The results are shown in Figs. 8 and 9.

Figure 8(a) reveals the influence of the normal load on the residual stress in the rolling (x) direction. Under the pure rolling condition (Q/P = 0) the residual stress in the rolling direction on the surface is zero, and the peak value is located in the range z/a = 0.7 – 1.0 depending on the p₀/k ratio. Evidently, under pure rolling contact the residual stress in the rolling (x) direction is compressive below the contact surface and zero on the contact surface. We confirm the influence of the normal load on the residual stress in axial (y) direction in Fig. 8(b). The location of the peak value of the residual stress in the axial (y) direction is about z/a = 0.6 and is almost independent of the p₀/k ratio. When p₀/k = 4.0 the axial residual stress, (σₓ), is zero on the surface. When the normal load becomes larger, the axial residual stress, (σₓ), on surface increases in compression. Both the distribution pattern and peak value of the residual stress in the rolling (x) direction differ from those of the residual stress in the axial (y) direction. At small p₀/k ratio in the pure rolling case, the residual stress in the axial (y) direction, (σᵧ), is slightly smaller than that in the rolling direction, (σₓ). When the p₀/k ratio becomes large, residual stress in axial direction, (σᵧ), exceeds that in the rolling...
direction, \( \sigma_0 \). From Figs. 8(a) and 8(b), it is noted that the normal load has a significant influence on the magnitude of the residual stresses and the residual stress zone size. Figure 8(c) exposes the influence of normal load on the residual shear strain rate, \( \Delta \gamma_{ab} \), which is the increment of the residual shear strain per passage of load and is expressed in nondimensional form as \( \Delta \gamma_{ab} = G/k \). At \( p_0/k = 4.0 \), there is no cumulative residual strain accumulation, which is in agreement with Johnson’s shakedown prediction (Johnson, 1962). Under pure rolling contact, the residual shear strain rate, \( \Delta \gamma_{ab} \), is found positive representing a forward surface flow consistent with Merwin-Johnson and McDowell-Moyar predictions.

The influence of tangential force on the steady state residual stresses and residual shear strain rate are portrayed in Fig. 9. The normal load, \( p_0/k \), is selected as 4.5 and the tangential force, \( Q/P \), varies from \(-0.2\) to \(+0.2\). According to the coordinate system shown in Fig. 1, a positive \( Q/P \) ratio corresponds to the loading situation in a driving wheel and a negative \( Q/P \) ratio simulates of a driven wheel. Referring to Fig. 9, the tangential force has an influence on residual stresses in both axial and rolling directions; however this influence is small compared with the influence of the normal load. We observe that, except in a thin layer of depth \( 0.2a - 0.3a \), the tangential force does not influence the residual stress zone size. When the normal loads are the same, Fig. 9(a) and Fig. 9(b) displays that a negative tangential force produces larger residual stresses than a positive tangential force of the same magnitude. The influence of the tangential force on the residual shear strain rate is outlined in Fig. 9(c). Both negative and positive residual shear strain rates depended on the \( Q/P \) ratio. The residual shear strain rate is always positive, which corresponds to a forward surface displacement rate, when \( Q/P = 0 \) (pure rolling) or is negative (driven wheel). When \( Q/P \) ratio is higher than a critical value, the residual shear strain rate becomes negative, which will generate backward surface displacement rate according to Eq. (22).

Fig. 10 Comparisons of the relative surface displacement per passage of load predicted by different approaches (1070 steel \( p_0/k = 4.5 \))

The cumulative surface displacement rate, \( \delta \), is obtained by integrating the residual shear strain rate in the following way,

\[
\delta = \int_0^\infty \Delta \gamma_{ab} dz
\]  

(22)

A positive \( \delta \) corresponds to a forward surface flow according to the coordinate system, Fig. 1. The three models are compared in Fig. 10 where \( \delta \) is normalized by \( G/ka \). The two surface nonlinear kinematic model predicts that constant ratchetting rate persists after cycle 11.

Discussion

We note that the plasticity models play a dominant role on the predictions of surface displacement rate (Jiang and Sehitoglu, 1992a). For example, a two surface model predicts “forward” surface displacement for rolling line contact, while the Garud hardening rule (Garud, 1981) with nested multiple surfaces predicts “backward” surface displacement for the same case. The two surface model utilized by McDowell-Moyar differed somewhat from the one adopted in this study, and it also predicted “backward” flow for pure rolling (McDowell and Moyar, 1986, 1991). The main difference between these models is the backstress evolution, i.e., translation of stress surface center for non proportional loadings, and the description of the instantaneous plastic modulus variation. The difference among these models is not pronounced for proportional, and in particular uniaxial loadings. Detailed discussion on the influence of different plasticity models on the residual stresses and strain ratchetting behavior will be presented in a forthcoming paper (Jiang and Sehitoglu, 1992b).

Many of the multiple surface models, and the Armstrong-Frederick type models with the dynamic recovery term tend to overpredict the ratchetting rates particularly after many cycles. The exponential decay of the ratchetting rates and the possible cessation of ratchetting, gradual closure of hysteresis loops with increasing cycles, and smooth loop shapes for unloading and reloading as observed in experiments are currently being incorporated into models. These issues are the subject of a future study.

We note the recent work of Yu (1992) and Yu et al. (1993) on residual stress determination in rolling contact. They applied Zarka’s operator split technique (Zarka, 1980; Zarka and Navidí, 1986) and specified the translation of a “modified backstress” so that the elastic-plastic rolling contact problem can be treated similar to the elastic case. In Zarka’s approach, the total stress is obtained as a summation of “elastic stress” and “residual stress.” To produce exact solutions, the “residual stress” should satisfy the equilibrium equations (statically admissible) and homogeneous boundary conditions at
any time during elastic-plastic deformation. The backstresses are calculated via a linear relationship (due to the assumptions of linear hardening), and as a result, the “residual stress” may not satisfy the equilibrium and the homogeneous boundary conditions. Enforcing the boundary conditions at the end of rolling pass is not sufficient to meet the field equations.

Both the direct approach of Yu et al. and the proposed one inevitably violate some field equations, yet provide results in general agreement with the FEM predictions. Some differences are noticeable between the two methods due to the different assumptions used in each approach. Yu et al.’s direct approach is computational easier and faster, but is limited to materials of pure kinematic hardening with constant plastic modulus function. No strain ratchetting can be predicted. The proposed approach on the other hand can easily accommodate complex nonlinear plasticity models perceived to have a first order effect on the ratchetting. New results from finite element analyses and rolling contact experiments will provide a test bed to further check the capability of these approaches.

Conclusions

1. Based on the initial assumption of elastic stresses and the employment of a relaxation procedure, an analytical approach has been developed for the approximate determination of residual stresses and strains in rolling contact problems. For the line contact problems, the proposed method provides residual stresses in favorable agreement with the finite element as well as the experimental results.

2. Using a nonlinear kinematic hardening model and the proposed method, the residual shear strain rate, Δγ̇_{slr}, and the cumulative surface displacement rate are established. The residual shear strain rate is confirmed to increase with shear traction in a nonlinear manner. The proposed method predicts that a driven wheel sustains greater incremental plastic deformation than the driving wheel yielding lower fatigue lives for the driven wheel. This prediction agrees with the experimental observations. Moreover, the higher compressive residual stresses for the driven wheel is consistent with the enhanced ratchetting rates.

3. Tangential force plays a considerable role on the residual stresses in the 0.3a (a is the half width of the contact area) thick layer beneath the contact surface, but has only a modest influence on the residual stresses beyond this thin layer. The normal load primarily determines the subsurface residual stresses and the size of the subsurface plastic zone.

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References


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Die plastischen Verzerrungsdeviationen sind daher identisch mit den plastischen Verzerrungen:

\[ e_x'' = e_x'', \quad e_y'' = e_y'', \quad e_z'' = e_z''. \] (5.2)

Weiters nehmen alle diese Theorien an, daß während des plastischen Fließens die Änderungsgeschwindigkeit der plastischen Verzerrung (oder, was dasselbe ist, die Änderungsgeschwindigkeit der plastischen Verzerrungsdeviation) in jedem Zeitpunkt der augenblicklichen Spannungsdeviation proportional ist. Um die Gleichungen, welche diese Annahme ausdrücken, mit den Gleichungen für die Änderungsgeschwindigkeit der elastischen Verzerrung kombinieren zu können, schreiben wir sie in der Form

\[ 2G \dot{e}_x'' = \lambda s_x, \quad 2G \dot{e}_y'' = \lambda s_y, \quad 2G \dot{e}_z'' = \lambda s_z, \]

\[ G \dot{\gamma}_{yx}'' = \lambda \tau_{yx}, \quad G \dot{\gamma}_{zx}'' = \lambda \tau_{zx}, \quad G \dot{\gamma}_{xy}'' = \lambda \tau_{xy}. \] (5.3)

Hierin ist \( \lambda \) ein positiver, skalarer Proportionalitätsfaktor, welcher in der Regel von der Zeit \( t \) und von den Koordinaten \( x, y, z \) abhängt. Es wird später gezeigt werden, daß wir mit Hilfe der Fließbedingung diesen Proportionalitätsfaktor aus unseren Gleichungen eliminieren können.

Nach dem Hookschen Gesetz ist die Änderungsgeschwindigkeit der elastischen Verzerrungsdeviation gegeben durch

\[ 2G \dot{e}_x = \dot{s}_x, \quad 2G \dot{e}_y = \dot{s}_y, \quad 2G \dot{e}_z = \dot{s}_z, \]

\[ G \dot{\gamma}_{yx} = \dot{\tau}_{yx}, \quad G \dot{\gamma}_{zx} = \dot{\tau}_{zx}, \quad G \dot{\gamma}_{xy} = \dot{\tau}_{xy}. \] (5.4)

Indem wir die Gln. (5.3) und (5.4) kombinieren, erhalten wir die folgenden Beziehungen für die Änderungsgeschwindigkeit der Gesamtverzerrung:

\[ 2G \dot{e}_x = \dot{s}_x + \lambda s_x, \quad 2G \dot{e}_y = \dot{s}_y + \lambda s_y, \quad 2G \dot{e}_z = \dot{s}_z + \lambda s_z, \]

\[ G \dot{\gamma}_{yx} = \dot{\tau}_{yx} + \lambda \tau_{yx}, \quad G \dot{\gamma}_{zx} = \dot{\tau}_{zx} + \lambda \tau_{zx}, \quad G \dot{\gamma}_{xy} = \dot{\tau}_{xy} + \lambda \tau_{xy}. \] (5.5)

Diese Gleichungen gelten natürlich nur während des plastischen Fließens, während also, gemäß der Fließbedingung (4.5) \( \dot{J}_2 \) den konstanten Wert \( k^2 \) hat. Mit anderen Worten, die Gln. (5.5) gelten nur solange, als

\[ \dot{J}_2 = k^2 \quad \text{und} \quad \dot{J}_{2} = 0 \] (5.6)

ist. Differenzieren wir Gl. (4.3) nach der Zeit und setzen das Ergebnis in die zweite Gl. (5.6) ein, dann erhalten wir

\[ \dot{J}_2 = s_x \dot{s}_x + s_y \dot{s}_y + s_z \dot{s}_z + 2 \tau_{yx} \dot{\tau}_{yx} + 2 \tau_{zx} \dot{\tau}_{zx} + 2 \tau_{xy} \dot{\tau}_{xy} = 0. \] (5.7)

Gl. (5.7) kann dazu benutzt werden, um den Proportionalitätsfaktor \( \lambda \) aus den Spannungs-Verzerrungsbeziehungen (5.5) zu eliminieren. Die sich hieraus ergebenden Ausdrücke werden sehr vereinfacht, wenn wir zur Abkürzung setzen:

\[ \tilde{W} = s_x \dot{e}_x + s_y \dot{e}_y + s_z \dot{e}_z + \tau_{yx} \dot{\gamma}_{yx} + \tau_{zx} \dot{\gamma}_{zx} + \tau_{xy} \dot{\gamma}_{xy}. \] (5.8)

Die Größe \( \tilde{W} \) kann als die Arbeit interpretiert werden, welche die Spannungen pro Zeit- und Volumseinheit bei der Gestaltänderung des Körpers leisten (im Gegensatz zur Volumsänderung; s. Anhang zu diesem Kapitel). Diese
Interpretation ist jedoch für das folgende nicht von Bedeutung. Wesentlich ist nur, daß die Größe $\dot{W}$ berechnet werden kann, sobald die Spannungen und die Verzerrungsgeschwindigkeiten bekannt sind.

Um $\lambda$ aus den Gln. (5,5) zu eliminieren, multiplizieren wir die ersten drei Gln. (5,5) der Reihe nach mit $s_x, s_y, s_z$ und die letzten drei mit $2 \tau_{yz}, 2 \tau_{zx}, 2 \tau_{xy}$, und addieren. Unter Benützung der Gln. (5,8), (5,7), (4,3) und der Fließbedingung (4,5) erhalten wir schließlich

$$2G \dot{W} = 2 \lambda k^2. \quad (5,9)$$

Da $\lambda$ als positive Größe definiert wurde, zeigt Gl. (5,9), daß $\dot{W}$ während des plastischen Fließens positiv sein muß. Wenn der Wert von $\lambda$, welcher aus (5,9) folgt, in die Gln. (5,5) eingesetzt wird und die so erhaltenen Gleichungen nach den Änderungsgeschwindigkeiten der Spannungsdeviationen aufgelöst werden, dann ergeben sich die folgenden Spannungs-Verzerrungsbeziehungen:

$$\dot{s}_x = 2G \left( \dot{e}_x - \frac{\dot{W}}{2 k^2} s_x \right), \quad \dot{\tau}_{yx} = G \left( \dot{\gamma}_{yx} - \frac{\dot{W}}{k^2} \tau_{yx} \right),$$
$$\dot{s}_y = 2G \left( \dot{e}_y - \frac{\dot{W}}{2 k^2} s_y \right), \quad \dot{\tau}_{zx} = G \left( \dot{\gamma}_{zx} - \frac{\dot{W}}{k^2} \tau_{zx} \right), \quad (5,10)$$
$$\dot{s}_z = 2G \left( \dot{e}_z - \frac{\dot{W}}{2 k^2} s_z \right), \quad \dot{\tau}_{xy} = G \left( \dot{\gamma}_{xy} - \frac{\dot{W}}{k^2} \tau_{xy} \right).$$

Diese Beziehungen gelten im plastischen Bereich, das heißt solange als $J_2 = k^2$ und $\dot{W} > 0$ ist. Wenn ein Spannungszustand gegeben ist, welcher die Fließbedingung $J_2 = k^2$ erfüllt, und Verzerrungsgeschwindigkeiten, welche, zusammen mit dem gegebenen Spannungszustand ein positives $\dot{W}$ liefern, dann bestimmen die Gln. (5,10) die Änderungsgeschwindigkeit des Spannungsdeviators. Um die Spannungsgeschwindigkeiten selbst zu erhalten, müssen wir die Gln. (5,10) mit der Beziehung zwischen der Änderungsgeschwindigkeit der mittleren Normalspannung und der Änderungsgeschwindigkeit der mittleren Dehnung kombinieren. Da der plastische Anteil der mittleren Dehnung als gleich Null angenommen wurde, erhalten wir diese Beziehung durch Differenzieren der letzten Gl. (3,2) nach der Zeit. Unter Benützung der Definition der mittleren Normalspannung [Gl. (1,5)] und der mittleren Dehnung [Gl. (2,3)] schreiben wir das Ergebnis dieser Differenziation in der Form

$$\dot{s} = 3K \dot{e}. \quad (5,11)$$

Da $\dot{\sigma}_x = \dot{s}_x + \dot{s}$ ist usw. [s. Gl. (1,9)], können die Spannungsgeschwindigkeiten $\dot{\sigma}_x, \dot{\sigma}_y, \dot{\sigma}_z$ mittels der Gln. (5,10) und (5,11) gefunden werden.

Im elastischen Bereich ($J_2 < k^2$) und für Entlastung aus einem Spannungszustand an der Fließgrenze ($J_2 = k^2$, aber $\dot{W} < 0$), tritt an Stelle der Gln. (5,10)