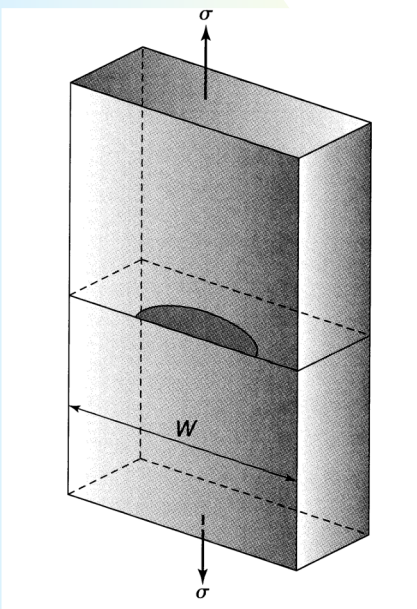
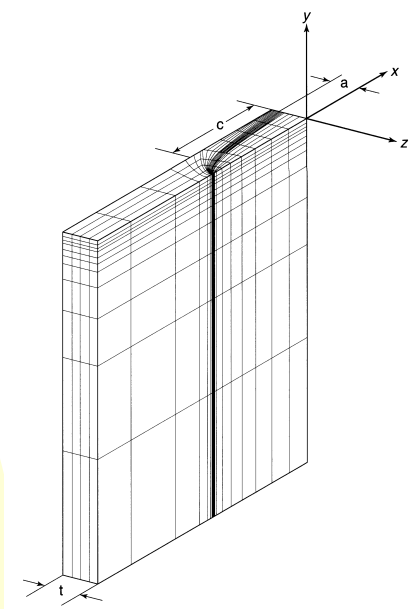
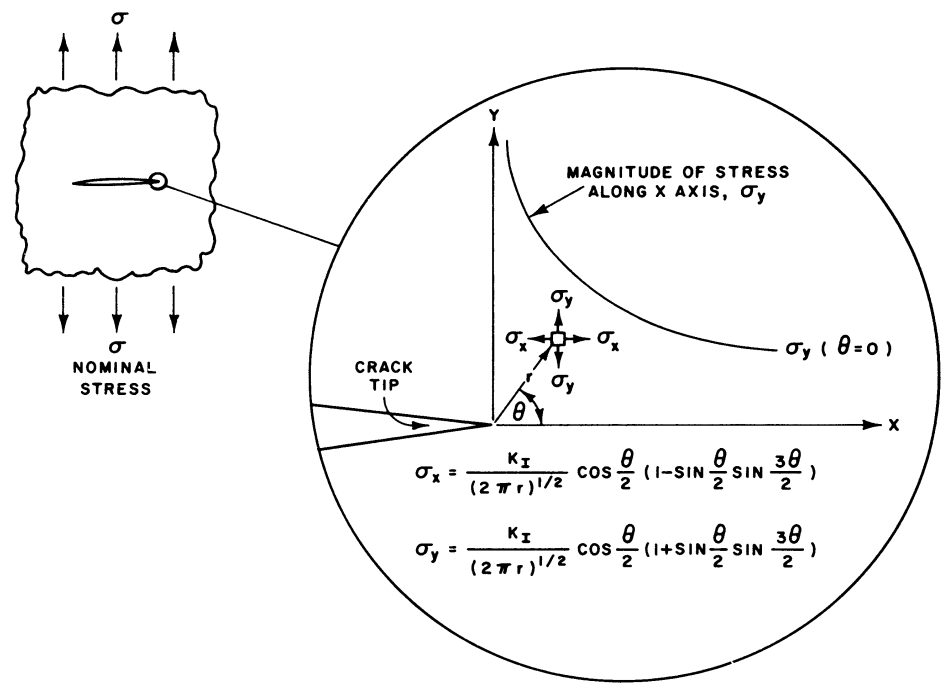
Stress-intensity factors

Sanford: pgs. 31-35, 51-60
Anderson: sections 2.6, A2.3







More on K_I factors

$$\sigma_{yy}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{yy}(\theta)$$

$$\sigma_{xx}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{xx}(\theta) + T$$

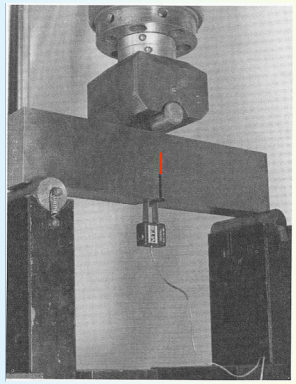
$$\tau_{xy}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{xy}(\theta)$$

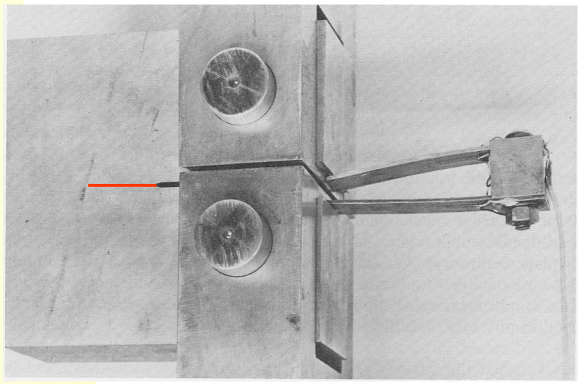
E and v <u>DO NOT</u> appear in these equations (they appear when loading is prescribed by enforced displacements)

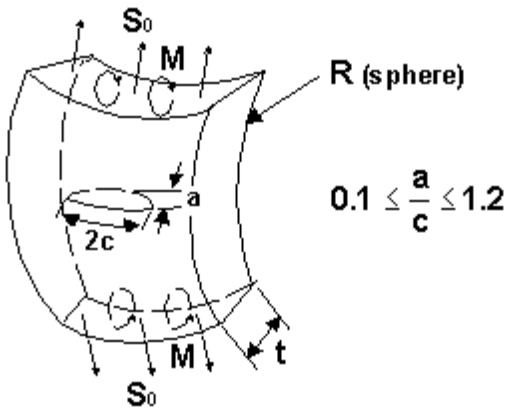
- **Independent of the specimen-structure geometry, crack length, type of loading (tension *vs*. bending), the crack tip stresses in *EVERY* case have these asymptotic values as *r* becomes "small".**
  - **At *r v. small*, the first term, $1/\sqrt{r}$, is so huge that none of the other terms matter**
  - **At larger *r*, the constant *T*-stress term becomes very important for σ_{xx} (σ_{yy} and τ_{xy} have no constant term – <u>very important</u> !)**
  - **At still larger distances, the non-singular $O(r,\theta)$ terms become most important, reflecting the "far field" loading (*e.g*. a bending stress, uniform tension, etc.)**

More on K_I factors

- All differences in specimen-structure geometry, crack size, loading type (tension *vs.* bending, thermal, etc.) appear only through the scalar stress-intensity factor (SIF), K_I, and the *T*-stress
- This is truly remarkable!
  - All components of the stress & strain (tensors) at each point (r,θ) very near the crack front, *and*
  - Two displacements (u,v) at each point (r, θ) very near the crack front
  - Are ***UNIQUELY*** determined by two scalar values (K_I and the T)







Middle Crack, M(T), Problem (1)

- Finite length crack located in an *infinite* body, loaded remotely by a uniform tensile stress

$$K_I \propto stress \times \sqrt{L}$$

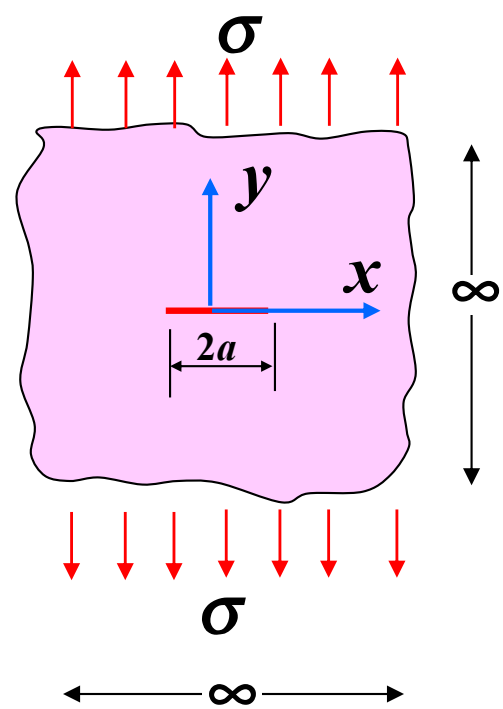
- Only "length" quantity in the model is crack length *a*
- Only load is the remote stress
- Then, only possible form is

$$K_I \propto \sigma \times \sqrt{a}$$

(dimensional analysis)

- Analytical (closed-form) solution shows that

$$K_I = 1.0 \times \sigma \sqrt{\pi a}$$



Plane stress or plane strain model.
Crack front is straight thru thickness

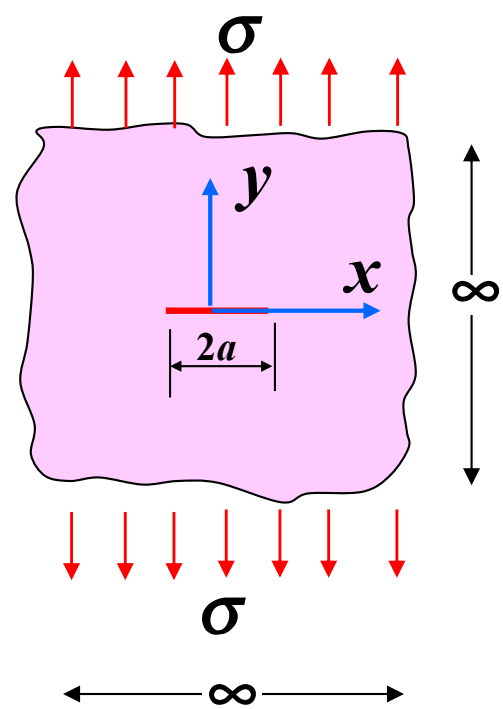
Middle Crack, M(T), Problem (2)

$$K_I = \sigma\sqrt{\pi a}$$

$$\sigma_{xx}(r,\theta) = \sigma\sqrt{\frac{a}{2r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{yy}(r,\theta) = \sigma\sqrt{\frac{a}{2r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy}(r,\theta) = \sigma\sqrt{\frac{a}{2r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$$



- **Suppose**

$$\sigma = 25\text{ ksi} \quad a = 1.0\text{ in.}$$

- **Then**

$$@\, r = 0.1\text{ in.},\ \sigma_{yy}(\theta = 0) = 56\text{ ksi}$$

$$@\, r = 0.05\text{ in.},\ \sigma_{yy}(\theta = 0) = 79\text{ ksi}$$

$$@\, r = 0.004\text{ in. } (100\mu\text{m}),\ \sigma_{yy}(\theta = 0) = 283\text{ ksi}$$

If σ_0=56 ksi, then yielded region just ahead of crack tip is about 0.1 in in size

How large is 100 um compared to microstructures of typical ferritic steels, Al alloys, Ti alloys?

Middle Crack, M(T), Problem (3)

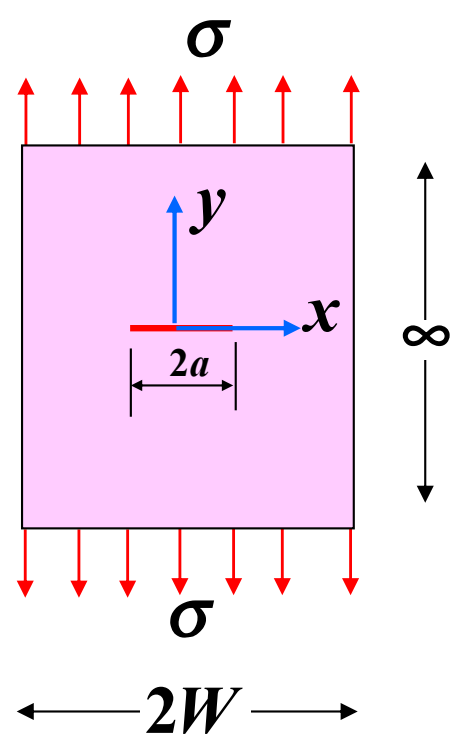
- Finite length crack located in an finite width, but *infinite* height body, loaded remotely by a uniform tensile stress

$$K_I \propto stress \times \sqrt{L}$$

- Now have 2 *length* quantities in the model: crack length *a* and panel width *W*
- Only load is the remote stress
- Then, only possible form is

$$K_I \propto \sigma \times \sqrt{a} \times \underbrace{F\left(\frac{a}{W}\right)}_{\text{non-dimensional > 1.0 in this case}}$$

(dimensional analysis)

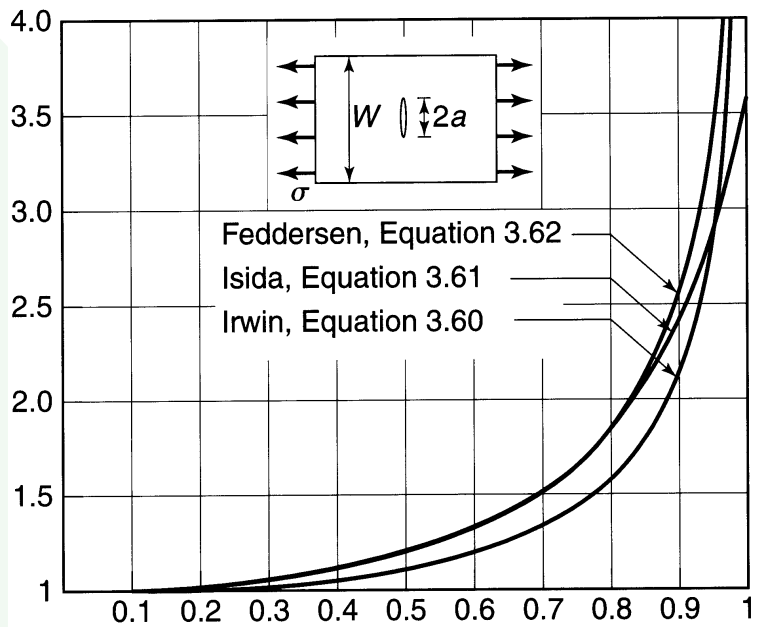


Plane stress or plane strain model.
Crack front is straight thru thickness

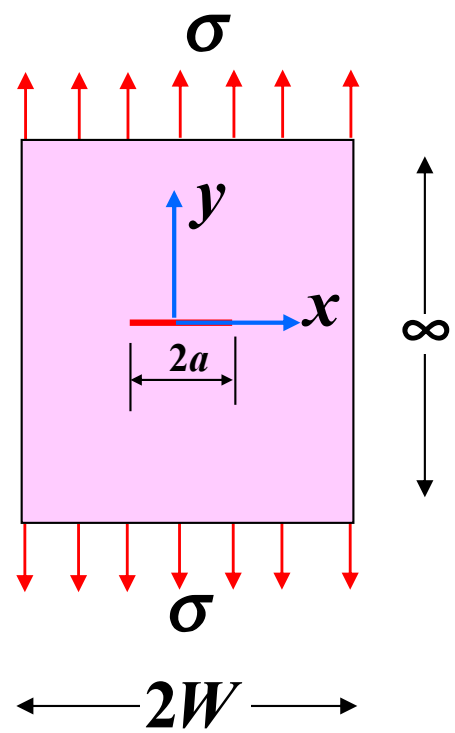
Middle Crack, M(T), Problem (4)

- **Accurate numerical solutions show that (earliest approximate solution by Irwin in 1957)**

$F(a/W)$



$2a/W$



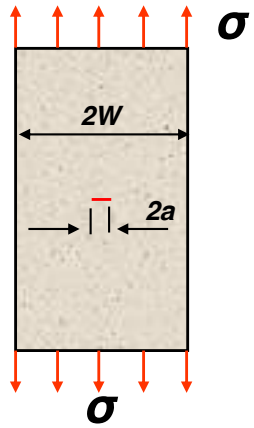
Plane stress or plane strain model. Crack front is straight thru thickness

$$K_I = \sigma \times \sqrt{\pi a} \times F\left(\frac{a}{W}\right)$$

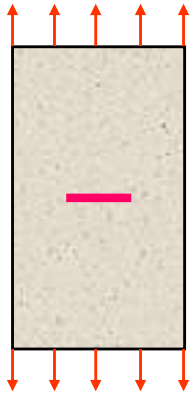
Note: **Handbook solution uses total panel width as *W*, not 2*W* as in Anderson**

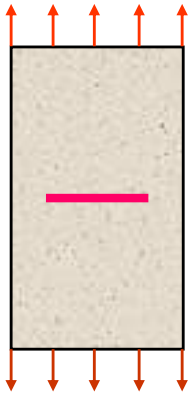
Middle Crack, M(T), Problem (5)

$$K_I = \sigma \times \sqrt{\pi a} \times F\left(\frac{a}{W}\right)$$



- For "small" *a*/*W*:
  - The crack behaves as if is in a nearly infinite size structure and *F*(*a*/*W*) goes to 1 as *a*/*W* goes to 0 (the real infinite body solution)
  - The axial stresses decrease rapidly away from the crack tips to very near the remotely applied stress level (because the area reduction from the crack is so small)
  - In the infinite body, the net section stresses do reach the remotely applied stress far from the tips!
  - *T*-stress is ≈ -1.0 × *σ* for (*a*/*W* < 0.1)



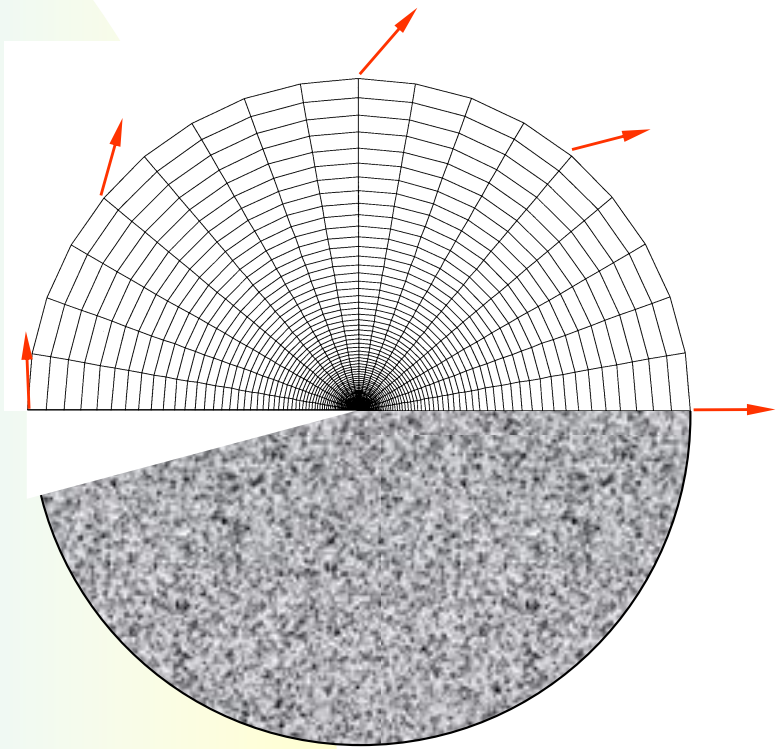


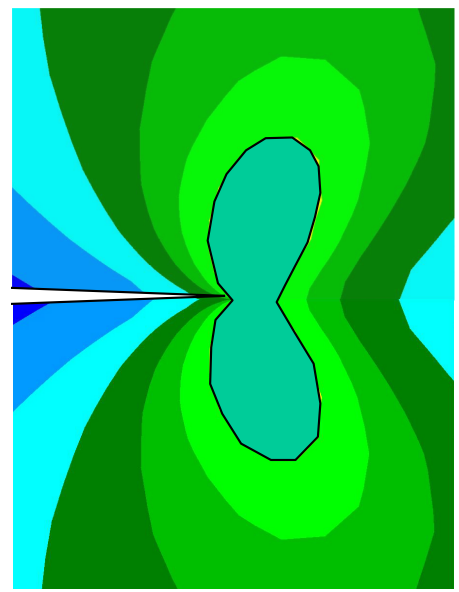
- For "large" *a*/*W* :
  - The average net section stress is a multiple of the remote stress
  - The crack front stress field interacts strongly with the traction free vertical edge
  - *F*(*a*/*W*) increases rapidly as *a*/*W* goes to 1.0
  - *T*-stress becomes more negative ≈ -1.5 × *σ*

Crack tip plasticity

Sanford: Chapter 6

Anderson: 2.8-2.9, Chapter 3 (early part)



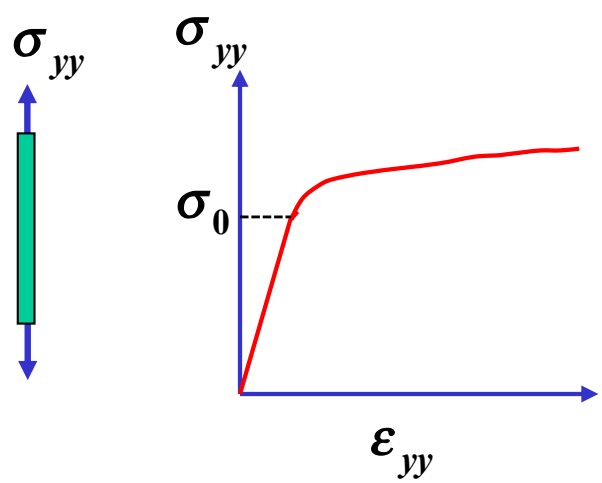


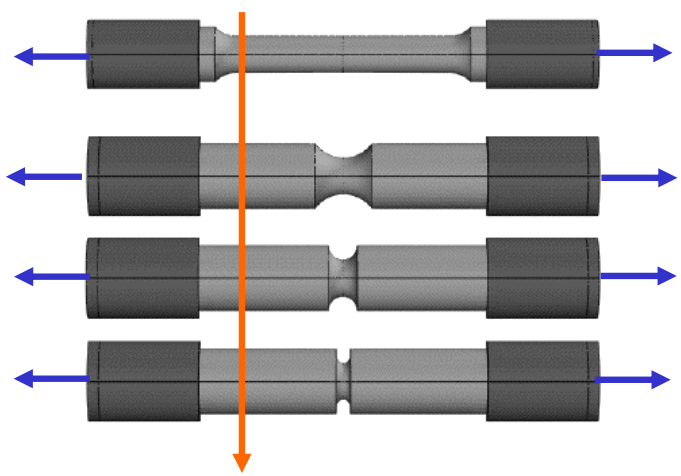
Metal plasticity (1)

- Simple uniaxial stress-strain curves
  - Idealized to have $\sigma_{xx} = \sigma_{zz} = 0$
  - Only non-zero stress is σ_{yy}
  - Round, notched tensile specimens introduce tensile stresses σ_{xx} and σ_{zz}
- The ***mean*** stress component does not contribute to yielding in multi-dimensions
- von Mises stress (a scalar) describes conditions for yielding under multi-axial loading to reflect no contribution from mean stress

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$$

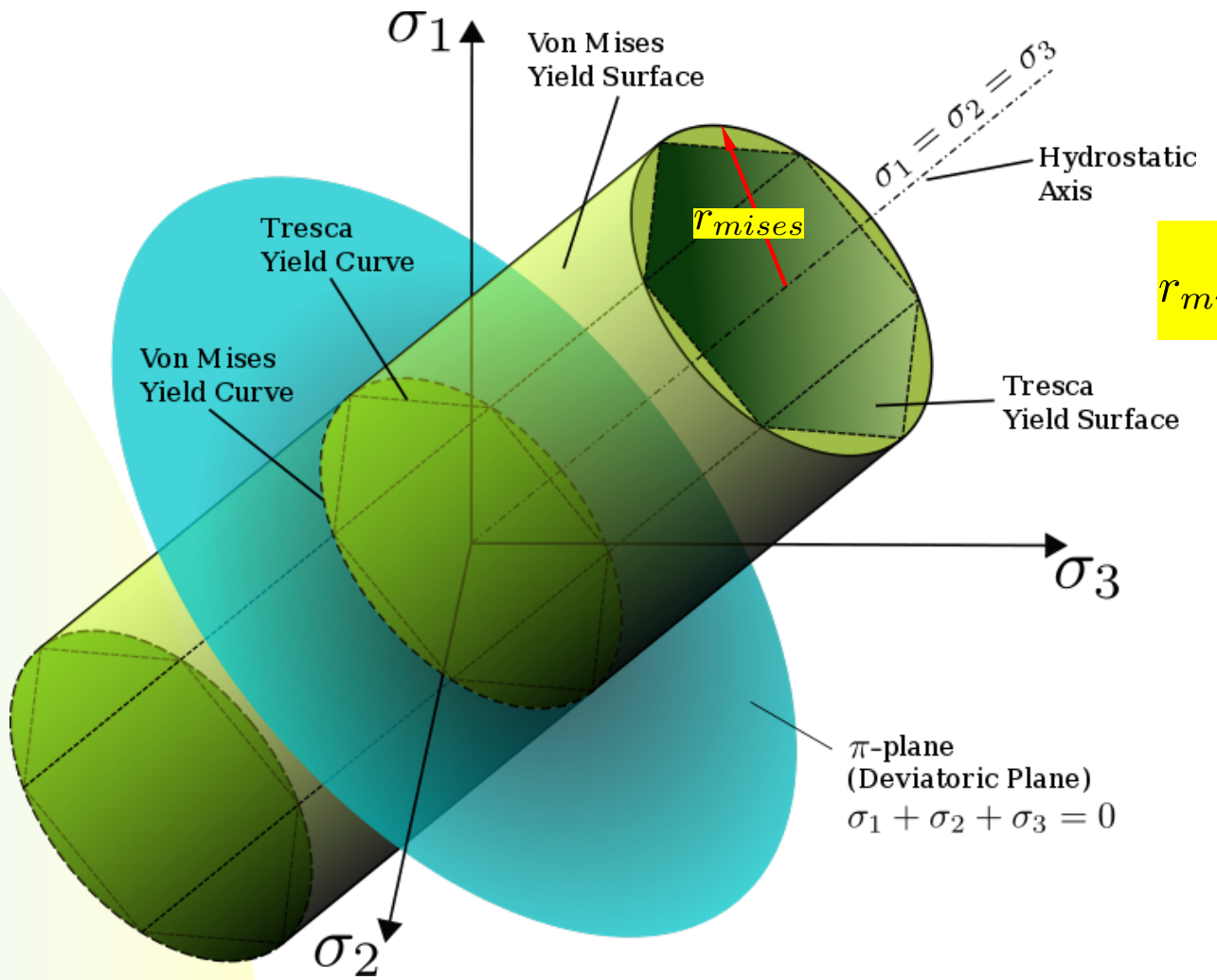
- Yield occurs when σ_e first reaches σ_0 under increased loading





Increasing mean stress with geometric severity of reduced section

Yield surfaces



$$r_{mises} = \sqrt{\frac{2}{3}}\sigma_0$$

http://en.wikipedia.org/wiki/Von_Mises_yield_criterion

Metal plasticity (2)

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}$$

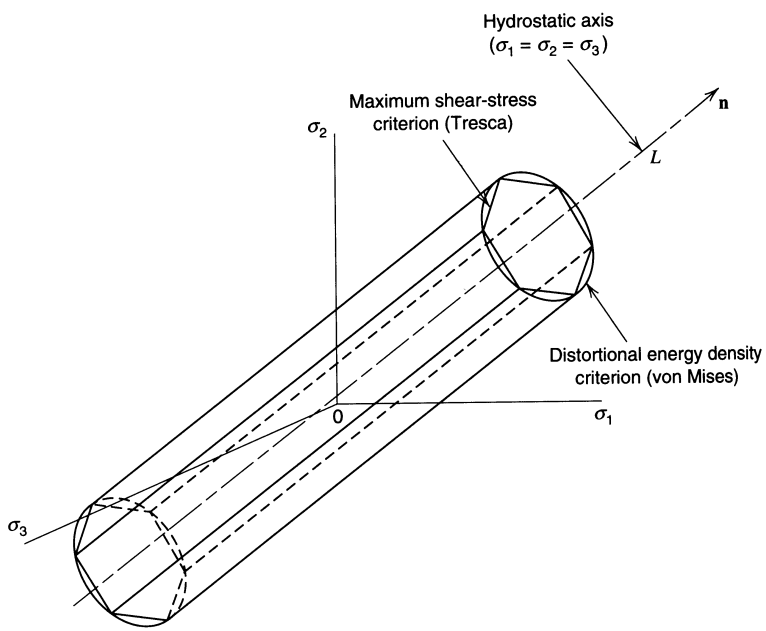


Figure 4.9 Yield surface in principal stress space.

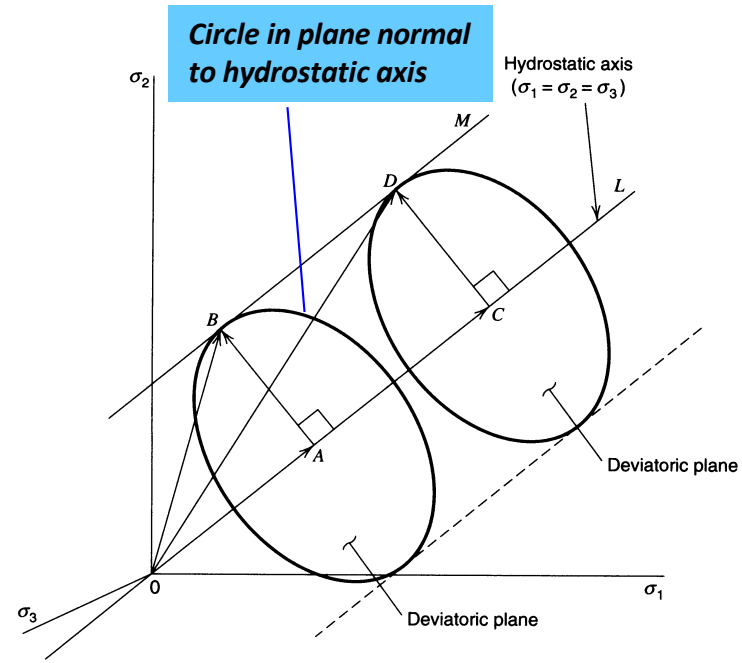
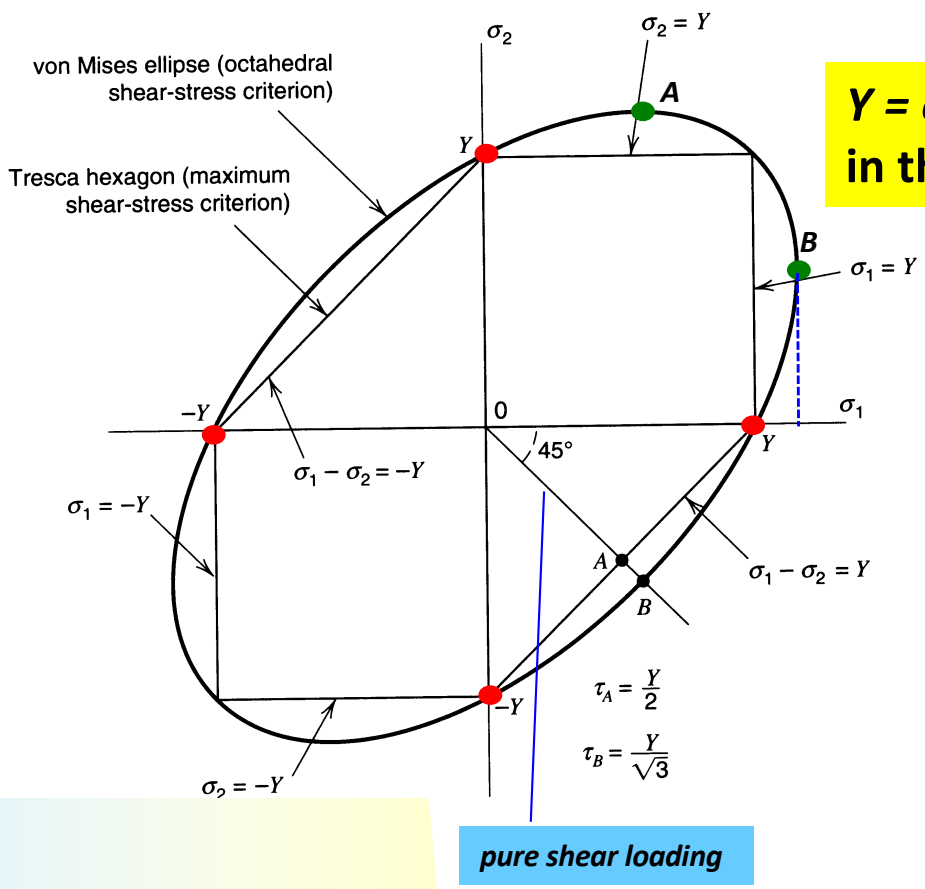


Figure 4.11

Boresi et al., Adv. Mechs. Matls. 5th Ed.

Metal plasticity (3)

- **For plane stress, the yield surface is an ellipse in the $\sigma_1 - \sigma_2$ plane ($\sigma_3 = 0$)**



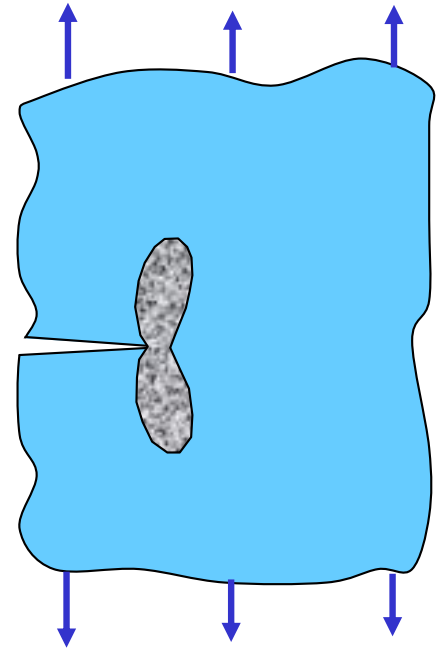
***$Y = \sigma_0$* in this figure**

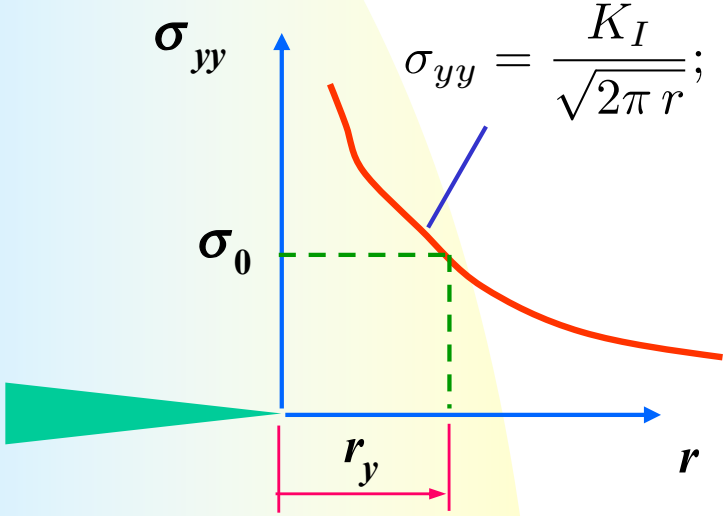
$$\sigma_e = \sigma_0 = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2}$$

Locate the values of σ_1 & σ_2 at points *A*, *B* shown. These are maximum attainable values which also satisfy the yield criterion for plane-stress conditions.

Contained plasticity

- **Small-Scale Yielding (SSY): crack front plastic zone is well-contained inside a surrounding large region of linear-elastic material**
- **Irwin estimate of plastic zone size on ligament**
  - **1-D model of plasticity ($\sigma_{xx} = \sigma_{zz} = 0$)**
  - **Only the opening mode stress considered**





$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi\, r}}; \quad \theta = 0$$

$$\sigma_0 = \frac{K_I}{\sqrt{2\pi\, r_y}}$$

$$r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_0}\right)^2 \longrightarrow r_y = \alpha\left(\frac{K_I}{\sigma_0}\right)^2$$

Non-dimensional factor α that varies with assumptions about plane-stress, plane-strain or something in-between, and material strain hardening

Units of length. This quantity appears everywhere in nonlinear fracture mechanics!

Continuum Plastic Zones (1)

- **Write principal stresses ahead of sharp crack tip for K_I field**

$$\sigma_1(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\right]$$

$$\sigma_2(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\right]$$

$$\sigma_3(r,\theta) = \nu\left(\sigma_1+\sigma_2\right) = 2\nu\frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right) \quad \text{(plane strain)}$$

$$\sigma_3(r,\theta) = 0 \quad \text{(plane stress)}$$

Plane Strain

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_1-\sigma_3\right)^2+\left(\sigma_2-\sigma_3\right)^2}$$

$$2\sigma_0^2 = \frac{K_I^2}{2\pi r_p}\left[\frac{3}{2}\sin^2\theta + \left(1-2\nu\right)^2\left(1+\cos\theta\right)\right]$$

Plane Stress

$$\sigma_e = \sigma_0 = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_1-\sigma_2\right)^2+\sigma_1^2+\sigma_2^2}$$

$$2\sigma_0^2 = \frac{K_I^2}{2\pi r_p}\left[1+\frac{3}{2}\sin^2\theta+\cos\theta\right]$$

Continuum Plastic Zones (2)

$$2\sigma_0^2 = \frac{K_I^2}{2\pi r_p}\left[\frac{3}{2}\sin^2\theta + (1-2\nu)^2(1+\cos\theta)\right]$$

Plane Strain

$$2\sigma_0^2 = \frac{K_I^2}{2\pi r_p}\left[1 + \frac{3}{2}\sin^2\theta + \cos\theta\right]$$

Plane Stress

- **Solve each eqn for r_p. This defines the distance from the crack tip along the ray (θ) at which the material is no longer yielding, *i.e.* if $r < r_p$, the material is undergoing plastic deformation**

$$r_p(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_0}\right)^2\left[\frac{3}{2}\sin^2\theta + (1-2\nu)^2(1+\cos\theta)\right]$$

Plane Strain

$$r_p(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_0}\right)^2\left[1 + \frac{3}{2}\sin^2\theta + \cos\theta\right]$$

Plane Stress

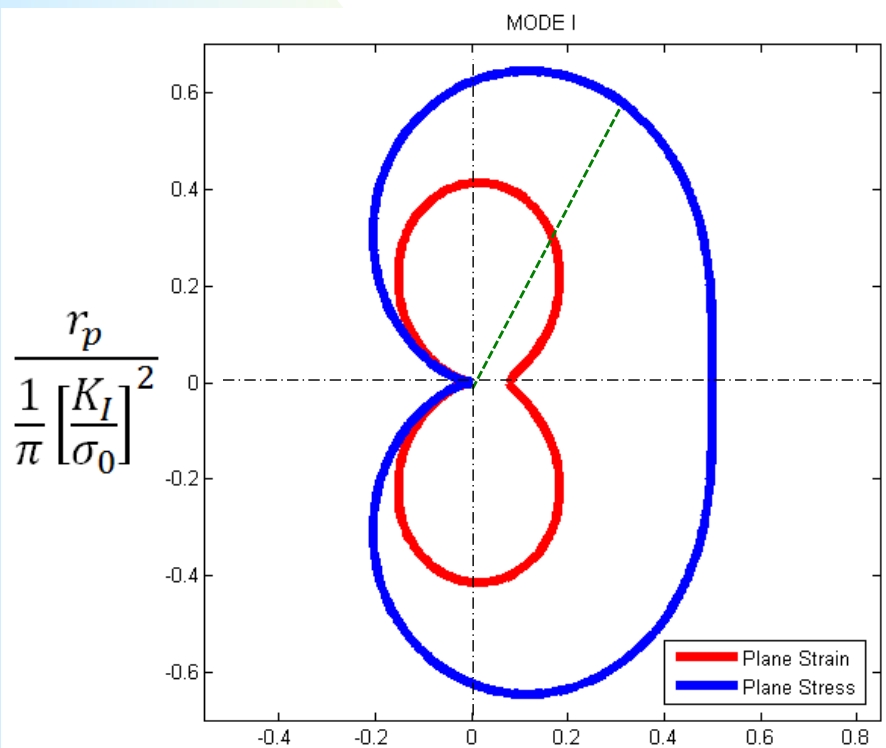
Continuum Plastic Zones (3)

$$r_p(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_0}\right)^2\left[\frac{3}{2}\sin^2\theta + (1-2\nu)^2(1+\cos\theta)\right]$$

Plane Strain

$$r_p(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_0}\right)^2\left[1 + \frac{3}{2}\sin^2\theta + \cos\theta\right]$$

Plane Stress



Anderson 3rd, pg. 68

$$r_p(\theta = 0) = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_0}\right)^2(1-2\nu)^2 = \alpha_\varepsilon\left(\frac{K_I}{\sigma_0}\right)^2 \text{(plane strain)}$$

$$\text{for } \nu = 0.3, \quad \alpha_\varepsilon = \frac{1}{12.5\pi}$$

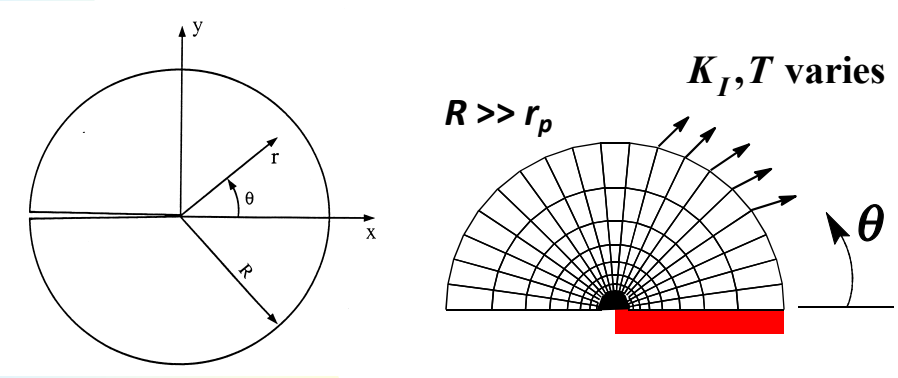
$$\text{for plane stress}, \quad \alpha_\sigma = \frac{1}{2\pi}$$

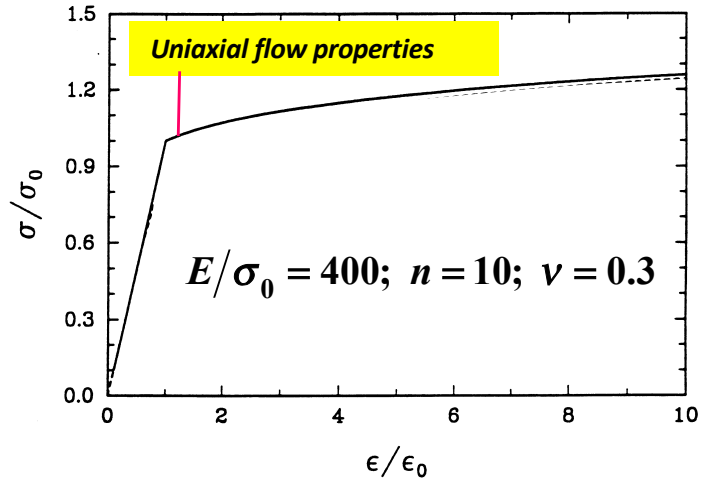
What is r_p for $\theta = 75^o$?

These equations and plots do not include re-distribution effects from plasticity

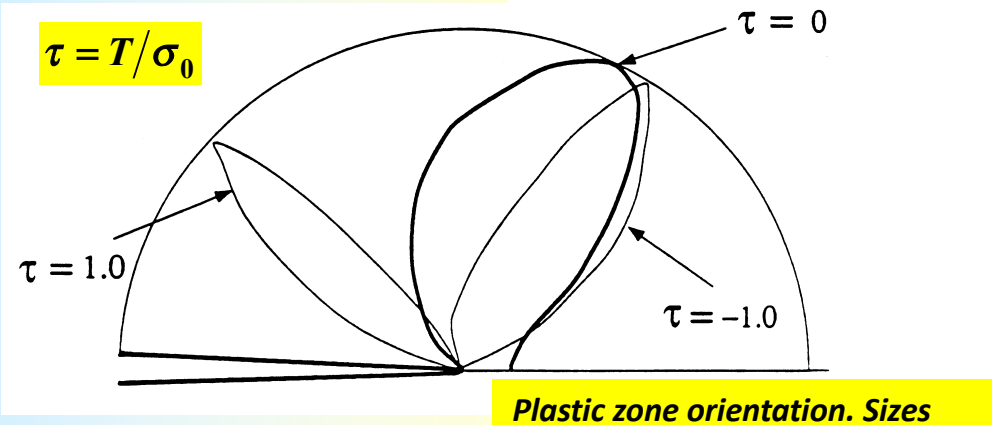
Contained Crack Front Plasticity

- Plane-strain, SSY FE solutions of Wang & Parks
- Effects of *T*-stress on real plastic zone sizes and shapes

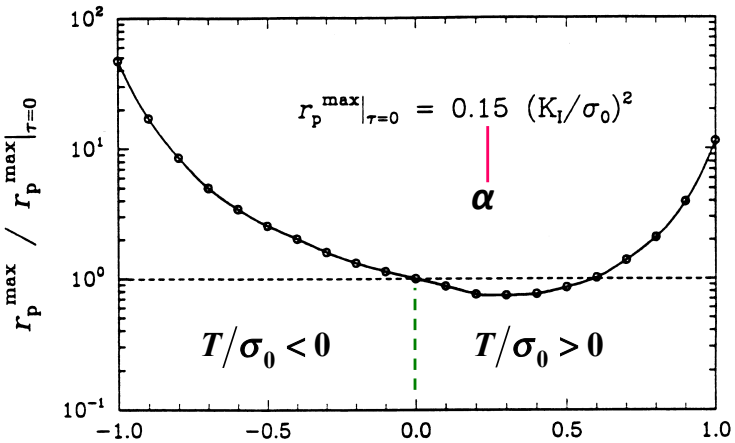




SSY, Plane-Strain, Mode I, FE Model



Plastic zone orientation. Sizes scaled to be identical



$\tau = T/\sigma_0$

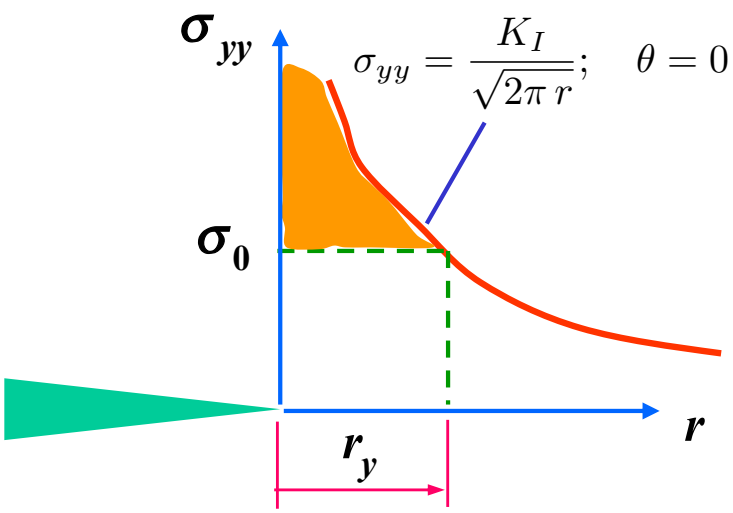
Plasticity Effect on K_I (1)

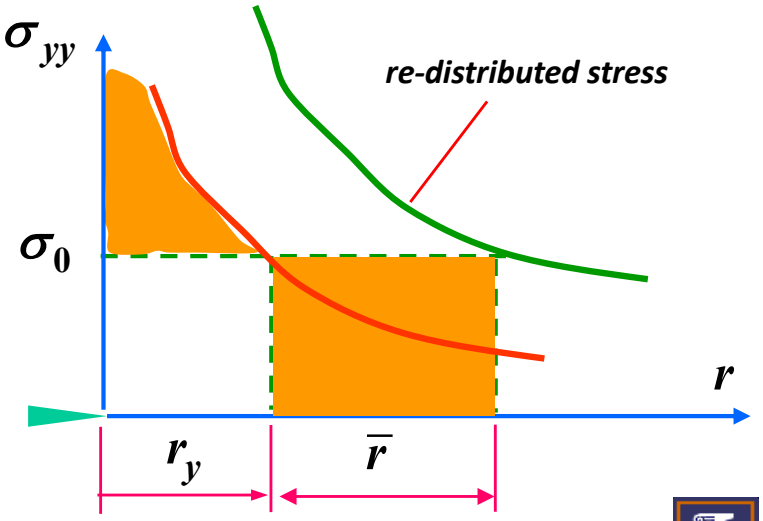
- Stress ahead of crack tip cannot exceed yield stress
- Force in shaded area must be ***re-distributed*** to material farther ahead of tip
- Increase σ_{yy} at locations $r > r_y$ above values given by $K_I/\sqrt{2\pi r}$
- For unit thickness, shaded area defines a force

$$F_u = (1) \cdot \int_0^{r_y} \sigma_{yy}(r)\, dr - \sigma_0 r_y$$

- We ***assume*** the re-distributed stress field has the same form as the original field
- Re-distribution increases the plastic zone size to

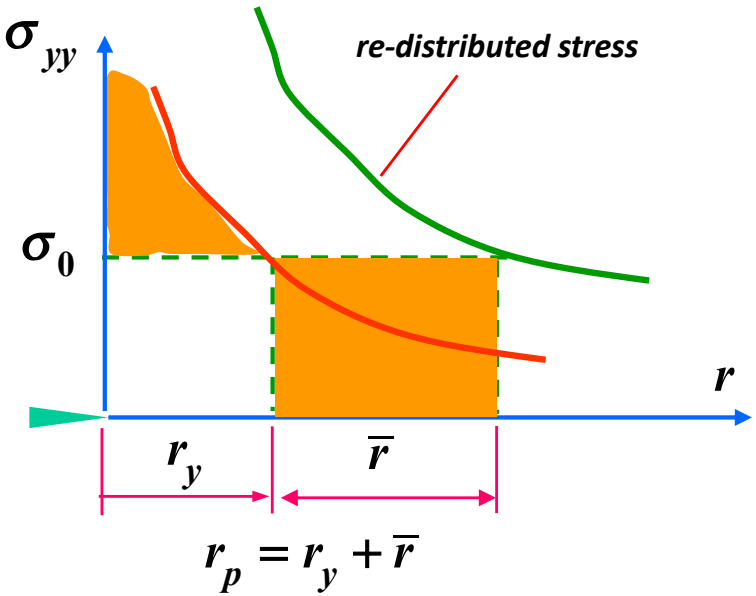
$$r_p = r_y + \bar{r}$$





Plasticity Effect on K_I (2)

- To 1st order, the shaded rectangular area must define a force to balance F_u



$$F_u = (1) \cdot \int_0^{r_y} \sigma_{yy}(r)\, dr - \sigma_0 r_y; \quad \sigma_0 \cdot \bar{r} \cdot (1) = F_u$$

$$\sigma_0 \cdot \bar{r} = \int_0^{r_y} \sigma_{yy}(r)\, dr - \sigma_0 r_y$$

$$\sigma_0 \cdot \underbrace{(r_y + \bar{r})}_{r_p} = \int_0^{r_y} \sigma_{yy}(r)\, dr = \int_0^{r_y} \frac{K_I}{\sqrt{2\pi r}}\, dr = \frac{K_I}{\sqrt{2\pi}} \int_0^{r_y} \frac{1}{\sqrt{r}}\, dr = \frac{K_I}{\sqrt{2\pi}} 2\sqrt{r}\Big|_0^{r_y}$$

$$\sigma_0 \cdot r_p = \frac{K_I}{\sqrt{2\pi}} 2\sqrt{r_y} = \frac{K_I}{\sqrt{2\pi}} 2 \frac{1}{\sqrt{2\pi}} \frac{K_I}{\sigma_0} \quad \rightarrow r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_0}\right)^2 = 2r_y$$

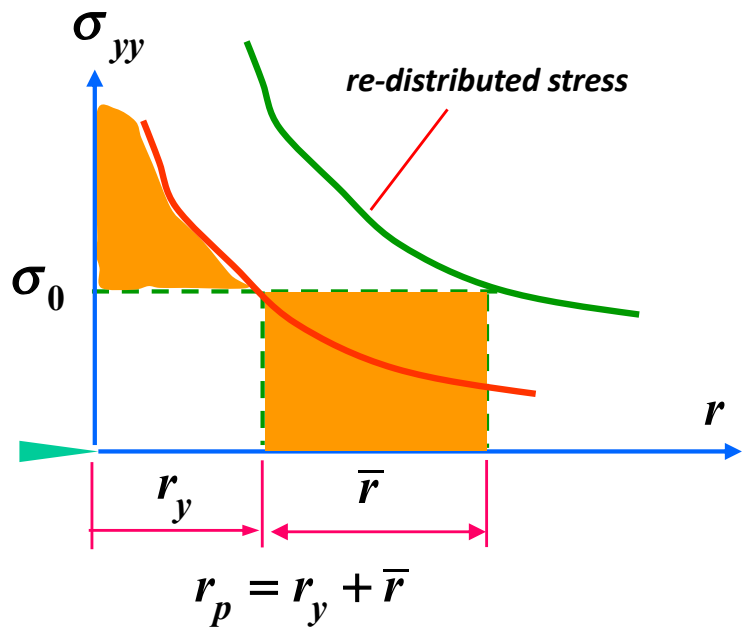
Effective Crack Length (a_{eff})

- The plastic zone size reflecting a simple, 1st order stress redistribution is 2 x the original estimate
- Irwin realized that the new stress distribution is thus the same as the linear-elastic field for a crack of length

$$a_{eff} = a + r_y$$

- The a_{eff} produces ≈ stress distribution for a real crack of length a, reflecting the 1st order plasticity effect
- This is called the ***Irwin plastic zone correction***
- a_{eff} is used in practice when the nominal applied stress w/o the crack is < ≈ 0.5 σ_0 (so that r_p is really small!)
- Then

$$K_I = F\left(a_{eff}\right)\sigma\sqrt{\pi a_{eff}}$$



$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_0}\right)^2 = 2r_y$$

Effective Crack Length (2)

$$K_I = F\left(a_{eff}\right)\sigma\sqrt{\pi a_{eff}} \qquad a_{eff} = a + r_y = a + \frac{1}{2\pi}\left(\frac{K_I}{\sigma_0}\right)^2$$

- **Now K_I appears on both sides of the equation**
- **Use a simple iterative scheme:**

(a) Compute K_I^{old} using $r_y = 0$

(b) Compute r_y using K_I^{old}

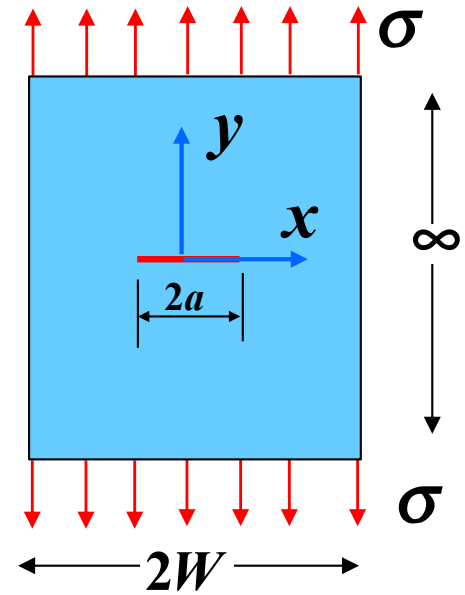
$$r_y = \alpha\left(\frac{K_I^{old}}{\sigma_0}\right)^2; \quad \alpha \doteq \frac{1}{2\pi} \text{ (plane stress)}; \; \alpha \doteq \frac{1}{6\pi} \text{ (plane strain)}$$

(c) Compute $a_{eff} = a + r_y$

(d) $K_I^{new} = F\left(a_{eff}\right)\sigma\sqrt{\pi a_{eff}}$

(e) if $\left|K_I^{new} - K_I^{old}\right| \leqslant tol$, $K_I = K_I^{new} \quad \Rightarrow$ Done

(f) otherwise, $K_I^{old} = K_I^{new}$, return to (b)



$$K_I = \left(\frac{2W}{\pi a}\tan\frac{\pi a}{2W}\right)^{1/2}\sigma\sqrt{\pi a}$$

Effective Crack Length (3)

- **For a large M(T) specimen, *F*(*a*/*W*) = 1.0 and the solution for effective crack length and K_I can be found in closed form**

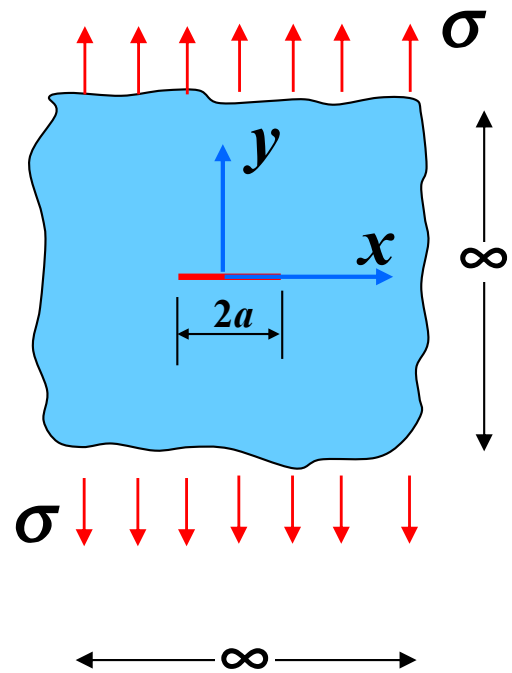
$$K_I = \sigma\sqrt{\pi a_{eff}}; \quad Note: \; F(a_{eff}) = 1.0$$

$$K_I = \sigma\sqrt{\pi\left\{a + \alpha\left(\frac{K_I}{\sigma_0}\right)^2\right\}}$$

$$K_I^{eff} = \frac{\sigma\sqrt{\pi a}}{\sqrt{1 - \alpha\pi\left(\frac{\sigma}{\sigma_0}\right)^2}}$$

$$\text{For plane stress,} \quad \alpha \doteq \frac{1}{2\pi}$$

$$K_I^{eff} = \sqrt{2}K_I \text{ when } \frac{\sigma}{\sigma_0} \to 1$$



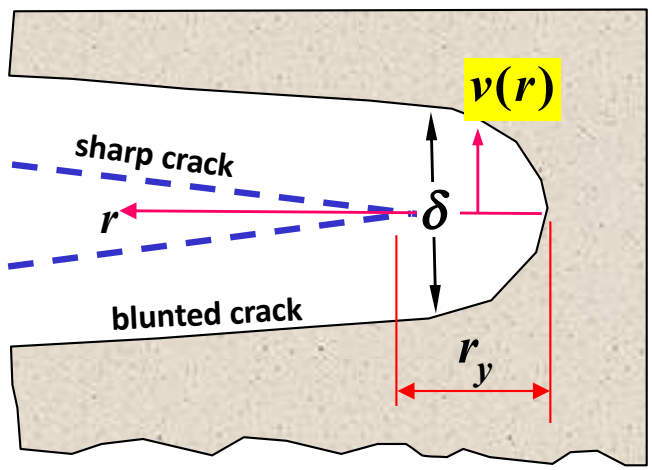
$$a_{eff} = a + r_y$$

$$r_y = \alpha\left(\frac{K_I}{\sigma_0}\right)^2$$

Suggestion: work out all steps

K_I – CTOD relation (1)

- Irwin's estimate of plastic zone size motivates a simple connection of CTOD with K_I
- Irwin proposed that the *blunted* opening at $r = 0$ is approximately the linear-elastic opening at $r = r_y$ on $\theta = \pi$ for a_{eff}



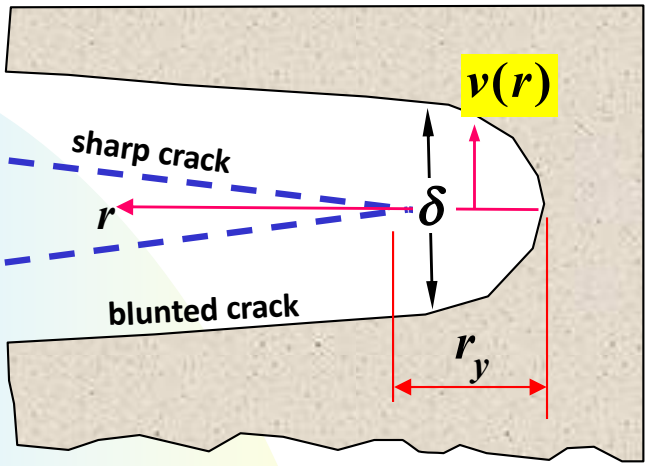
$$v(r) = \frac{\kappa + 1}{2G} K_I \sqrt{\frac{r}{2\pi}}; \quad \theta = \pi$$

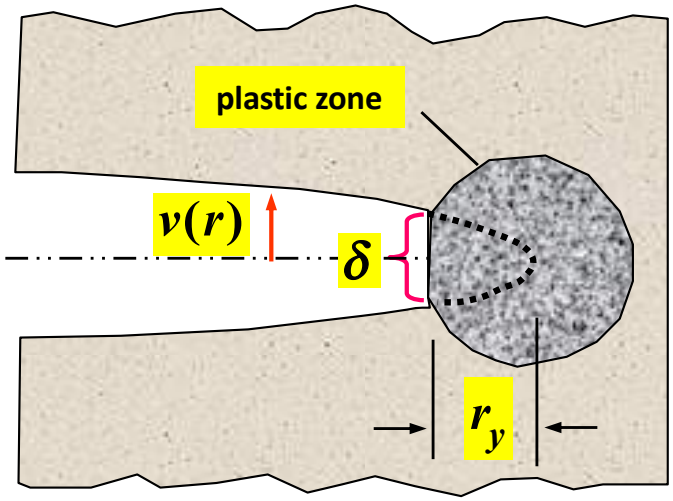
$$\kappa = 3 - 4\nu \quad (Plane\ Strain)$$

$$\kappa = \frac{3 - \nu}{1 + \nu} \quad (Plane\ Stress)$$

$$v(r) = \frac{4}{\pi} K_I \sqrt{\frac{r}{2\pi}}$$ ***Plane stress***

K_I – CTOD relation (2)





$$v(r) = \frac{4}{\pi} K_I \sqrt{\frac{r}{2\pi}}$$

Plane Stress

$$\delta = 2 \times v(r = r_y) = \frac{8}{E} K_I \sqrt{\frac{\frac{1}{2\pi}\left(\frac{K_I}{\sigma_0}\right)^2}{2\pi}}$$

$$\therefore \delta = \frac{4}{\pi} \frac{K_I^2}{\sigma_0 E} \Rightarrow$$

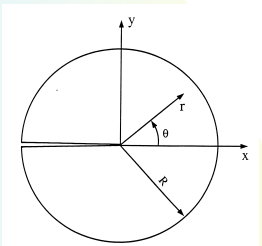
CTOD increases as K^2
(this is a nonlinear effect!)

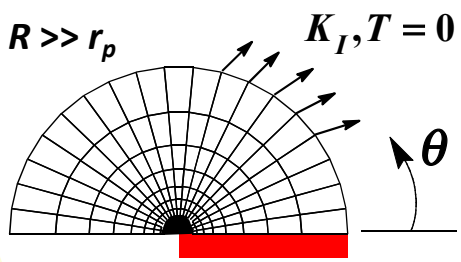
The actual coefficient (not 4/π) is found from nonlinear FE analyses. Waries with material strain hardening and 3-D effects.

Suggestion: re-derive for plane-strain conditions, verify physical units are correct.

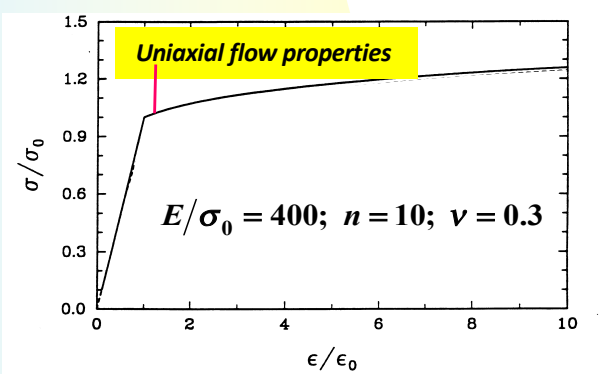
Similarity length scales (1)

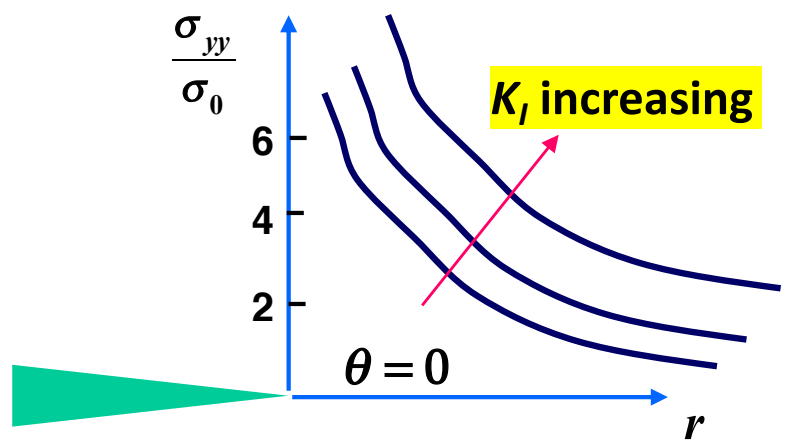
- Full-scale nonlinear finite element analyses of the plane-strain SSY model with T = 0 reveal complete features of the fields
- Consider 1st solutions based on small-strain deformation theory (crack tip remains sharp, *i.e.* nodal coordinates are never updated during analysis)





SSY, Plane-Strain, Mode I, FE Model





Stresses –> ∞ since tip remains sharp in the analysis even with plasticity, *i.e.* coordinates are <u>not updated</u> to deformed shape

Similarity length scales (2)

- **The analyses show that the nonlinear stresses have the form:**

$$\sigma_{ij} \propto \sigma_0 \underbrace{\left[\frac{r}{(K_I/\sigma_0)^2}\right]^{\zeta}} \underbrace{\tilde{\sigma}_{ij}\left(\theta, n, \nu, \frac{E}{\sigma_0}\right)}; \quad i,j = 1,\ 2$$

Non-dimensional distance, r*, ahead of crack tip. Essentially distance / plastic zone size. ζ varies with hardening exponent n

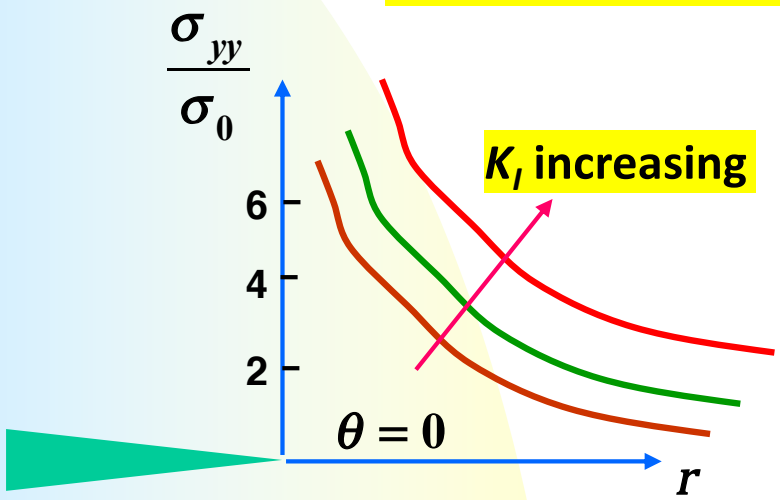
Non-dimensional angular function for each stress component. Varies with flow properties of the material

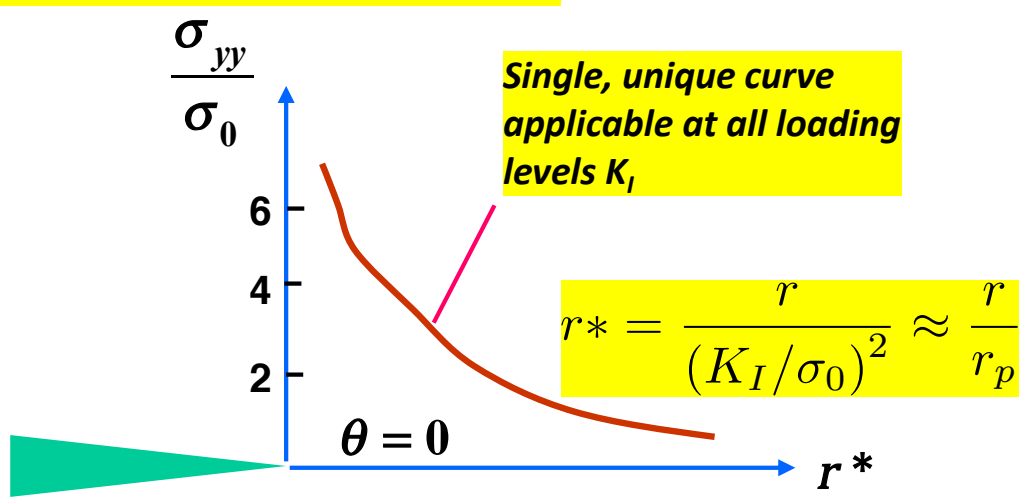
This is really the Hutchinson-Rice-Rosengren (HRR) field (1968) to be discussed later in course….

Similarity length scales (3)

- This is a simple power-law form. For a specific material and on $\theta = 0$, the opening mode stress becomes simply

$$\sigma_{yy} = \sigma_0 \left[\frac{r}{(K_I/\sigma_0)^2} \right]^{\varsigma} \hat{\sigma}_{yy} = \sigma_0 \times \ (r*)^{\varsigma} \times \ \hat{\sigma}_{yy}$$

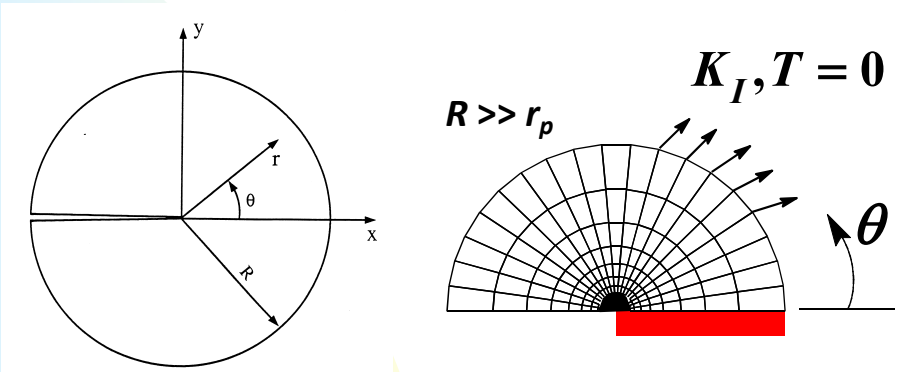




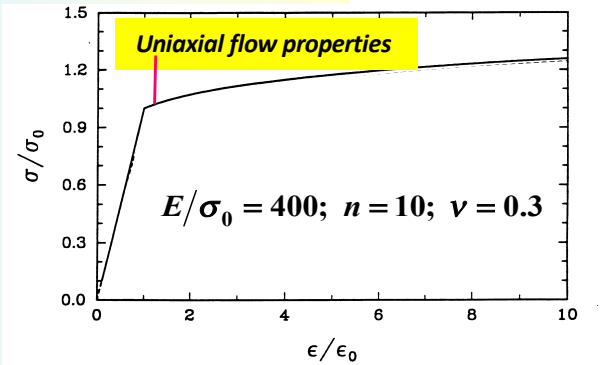
- This is an ***incredible*** result and <u>underlies much fundamental understanding of fracture mechanics</u>
- This is a consequence of ***self-similar SSY solutions***, *i.e.*, at every loading level the solution normalizes to a unique function

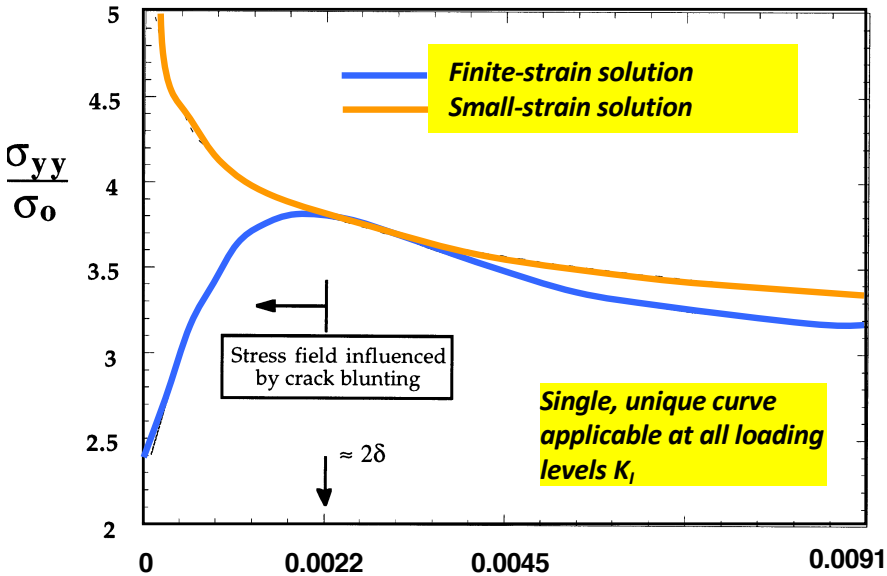
Similarity length scales (4)

- **If we repeat the SSY, plane-strain analyses using *finite strain* formulations to capture crack tip blunting, self-similar solutions are again found**



SSY, Plane-Strain, Mode I, FE Model





$$r* = \frac{r}{(K_I/\sigma_0)^2} \approx \frac{r}{r_p}$$

Dugdale strip-yield model (1)

- **Two key papers published in the early 1960s developed somewhat more sophisticated crack tip plasticity models (still using 1-D plasticity)**

Yielding of steel sheets containing slits

D. S. Dugdale
Engineering Department, University, College of Swansea, USA

Journal of the Mechanics and Physics of Solids
Volume 8, Issue 2 , May 1960 , Pages 100-104

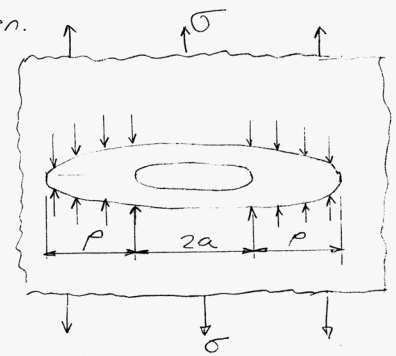
Abstract
Yielding at the end of a slit in a sheet is investigated, and a relation is obtained between extent of plastic yielding and external load applied. To verify this relation, panels containing internal and edge slits were loaded in tension and lengths of plastic zones were measured.

B. A. Bilby, A. H. Cottrell, and K. H. Swinden, The Spread of Plastic Yield from a Notch, Proc. R. Soc. Lond. A March 19, 1963 272:304-314; doi:10.1098/rspa.1963.0055

Abstract
A calculation is made of the length of plastic zone needed to accommodate a given plastic displacement at the root of a notch in a uniformly stressed solid. ….

Dugdale strip yield model (2)

Consider a large M(T) specimen.



Material a distance ρ beyond each physical crack tip is yielded, $\sigma_{yy} = \sigma_0$ (plane-stress)

The stresses $\sigma_{yy} = \sigma_0$ acting over ρ are holding the physical crack closed of a length $2a$.

Dugdale developed a procedure to find the distance ρ in terms of σ_0, $2a$, K_I.

Let: $K_\sigma = \sigma\sqrt{\pi(a+\rho)}$ } SIF for a crack of length $2(a+\rho)$ loaded by remote stress σ

Let: $K_\rho =$ K_I due to closing stresses $\sigma_{yy} = \sigma_0$ acting over ρ at each end of crack of length $2(a+\rho)$.
$K_\rho < 0$ due to direction of closing stresses

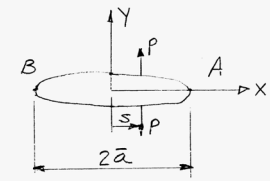
Dugdale argues that

$$K_\sigma + K_\rho \equiv 0 \quad @ \quad \bar{a} = a + \rho$$

when the closing stresses just close the crack completely over the length ρ, i.e., the crack tip singularity at $\bar{a}$ must vanish since there is no crack there.

Solution for K_ρ

Use K_I solutions for a pair of splitting forces applied to crack faces.



$$K_A = \frac{P}{\sqrt{\pi\bar{a}}}\sqrt{\frac{\bar{a}+x}{\bar{a}-x}}$$

$$K_B = \frac{P}{\sqrt{\pi\bar{a}}}\sqrt{\frac{\bar{a}-x}{\bar{a}+x}}$$

Let $P = \sigma_0\,dx$ (unit thickness)

Apply these over $s \le x \le \bar{a}$ in above configuration. Include the influence of each P on each crack tip. Due to symmetry, $K_A = K_B$.

$$\bar{K}_{A,B} = \frac{\sigma_0}{\sqrt{\pi\bar{a}}}\int_{x=s}^{x=\bar{a}}\left\{\sqrt{\frac{\bar{a}+x}{\bar{a}-x}} + \sqrt{\frac{\bar{a}-x}{\bar{a}+x}}\right\}dx$$

The solution is:

$$\bar{K}_I = 2\sigma_0\sqrt{\frac{\bar{a}}{\pi}}\cos^{-1}\left(\frac{s}{\bar{a}}\right)$$

In our Dugdale model, $s = a$, $\bar{a} = a+\rho$ and σ_0 is in the reverse direction

$$K_\rho = -2\sigma_0\sqrt{\frac{a+\rho}{\pi}}\cos^{-1}\left(\frac{a}{a+\rho}\right)$$

Then,

$$K_\sigma + K_\rho = 0$$

$$\sigma\sqrt{\pi(a+\rho)} = 2\sigma_0\sqrt{\frac{a+\rho}{\pi}}\cos^{-1}\left(\frac{a}{a+\rho}\right)$$

$$\cos^{-1}\left(\frac{a}{a+\rho}\right) = \left(\frac{\sigma}{\sigma_0}\right)\left(\frac{\pi}{2}\right)$$

Dugdale strip yield model (3)

$$\frac{a}{a+\rho} = \cos\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)$$

Solve for $a+\rho = a\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)$

Then,

$$\therefore K_I^{eff} = \sigma\sqrt{\pi(a+\rho)} = \sigma\sqrt{\pi a\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)}$$

We can compare the Dugdale estimate for the plastic zone size (ρ) with the Irwin estimate, $2r_y$

Expand cos( ) in a Taylor series and solve for ρ

$$\rho \approx \frac{\pi^2\sigma^2 a}{8\sigma_0^2} = \frac{\pi K^2}{8\sigma_0^2} = \frac{\pi}{8}\left(\frac{K}{\sigma_0}\right)^2 = \alpha_D\left(\frac{K}{\sigma_0}\right)^2$$

see footnote ↑ Dugdale

Compare with $r_p = 2r_y = 2\cdot\frac{1}{2\pi}\left(\frac{K}{\sigma_0}\right)^2 = \alpha_I\left(\frac{K}{\sigma_0}\right)^2$ ↑ Irwin

$\alpha_D = 0.392$ $\alpha_I = 0.318$

A further extension of this approach was published by Burdekin & Stone, J. of Strain Analysis, 1966. Their model is termed the "log-secant" formula

$$\therefore K_I^{eff} = \sigma_0\sqrt{\pi a}\sqrt{\frac{8}{\pi^2}\ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)\right]}$$

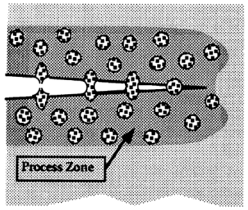
$$K_I^{eff} = \sigma_0\sqrt{\pi a}\sqrt{\frac{8}{\pi^2}\ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)\right]}$$

$$\cos\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right) \approx 1 - \frac{1}{2!}\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)^2$$

Applications of Dugdale Model

Dugdale type models are very often used to describe quantitatively the role of toughening mechanisms in otherwise brittle materials. Sect. 6.2.

The background brittle material is "toughened" by the ductile inclusions. These can be rubber, metal fibers, etc.



The SIF at the macroscopic crack is then

$$K_{tip} = K_{app} - K_{TM}$$

due to forces across crack surface

applied K from remote load w/o toughening mechanism

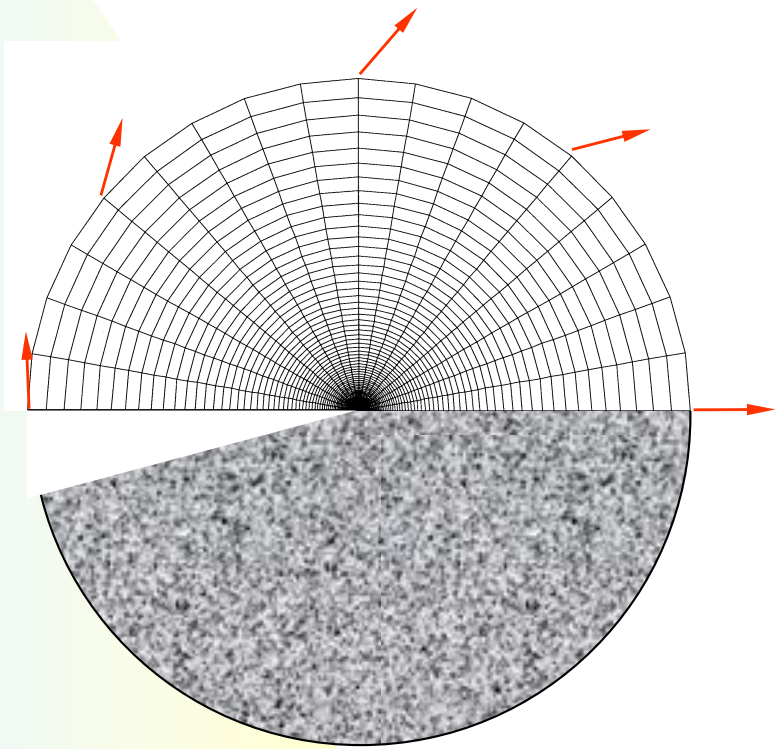
TABLE 6.1 Ceramics with enhanced toughness [39].

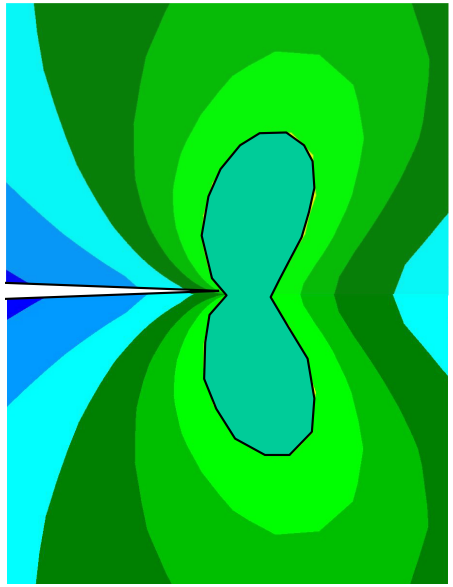
Toughening Mechanism	Material	Maximum Toughness, MPa $\sqrt{m}$
Fiber reinforced	LAS/SiC Glass/C SiC/SiC	~20 ~20 ~20
Whisker reinforced	Al_2O_3/SiC(0.2) Si_3N_4/SiC(0.2)	10 14
Ductile network	Al_2O_3/Al(0.2) B_4C/Al(0.2) WC/Co(0.2)	12 14 20
Transformation toughened	PSZ TZP ZTA	18 16 10
Microcrack toughened	ZTA Si_3N_4/SiC	7 7

For comparison, structural steels at very cold temperature have toughness 20-40 MPa $\sqrt{m}$

Plane strain induced constraint

Anderson: sections 2.8.1, 2.8.3

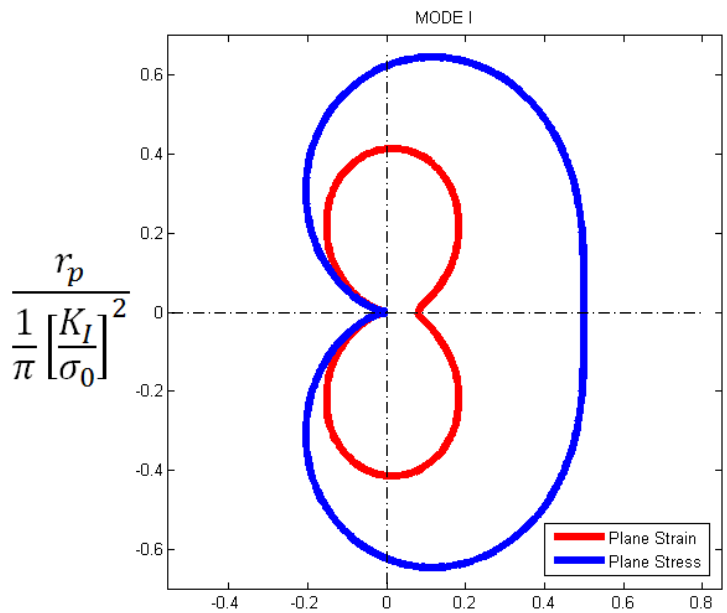




Metal plasticity

- The shapes of the plastic zones for plane stress and plane strain idealizations reveal key features
- At the same K_I value, the plane-strain plastic zone is smaller
- Implies higher ***mean*** stress and smaller ***deviatoric*** stress

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}$$



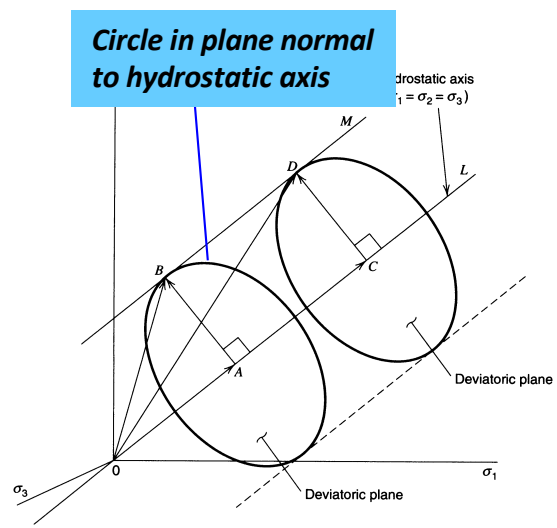


Figure 4.11

Plane-strain (1)

- **We can estimate the 1st order effect of the *constraint* suppressing plastic flow caused by the plane-strain conditions**
- **For simplicity, assume we have proportional loading at points ahead of the crack tip. Then**

$$\sigma_2 = n\sigma_1; \quad \sigma_3 = m\sigma_1$$
$$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \text{ (principal stresses)}$$

- **Insert these into the Mises yield criterion**

$$\sigma_0 = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}$$
$$2\sigma_0^2 = \left[(1-n)^2+(n-m)^2+(1-m)^2\right]\sigma_1^2$$
$$\text{or}$$
$$\frac{\sigma_1}{\sigma_0} = \left[1-n-m+n^2+m^2-nm\right]^{-1/2}$$

Any proportional stress field – not limited to crack tip fields

Plane-strain (2)

- **Now use the Mode I, plane-strain values for the principal stresses**

$$\sigma_1(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\right]$$

$$\sigma_2(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\right]$$

$$\sigma_3(r,\theta) = \nu\left(\sigma_1+\sigma_2\right) = 2\nu\frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)$$

$$n = \frac{\sigma_2}{\sigma_1} = \frac{1-\sin\frac{\theta}{2}}{1+\sin\frac{\theta}{2}}; \quad m = \frac{\sigma_3}{\sigma_1} = \frac{2\nu}{1+\sin\frac{\theta}{2}}$$

- **On θ = 0 (along the crack plane), $\sigma_{xx} = \sigma_{yy} = \sigma_1 = \sigma_2$**
- **These give n = 1, m = 2ν = 0.6 for ν = 0.3**
- **Then**

$$\frac{\sigma_1}{\sigma_0} = \left[1-n-m+n^2+m^2-nm\right]^{-1/2}$$

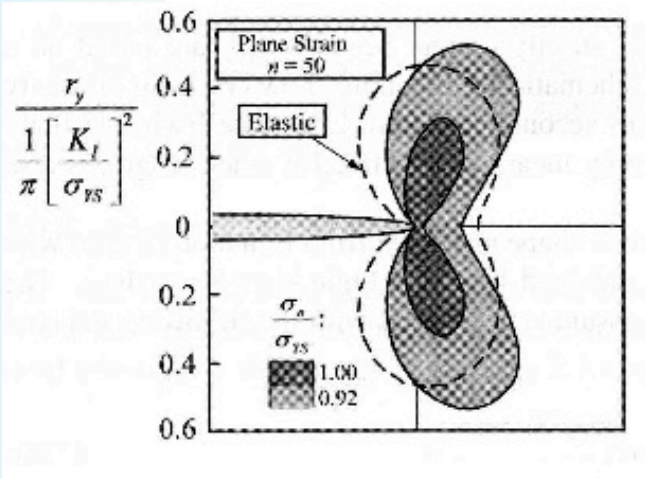
$$\text{with } n=1, \quad m=0.6$$

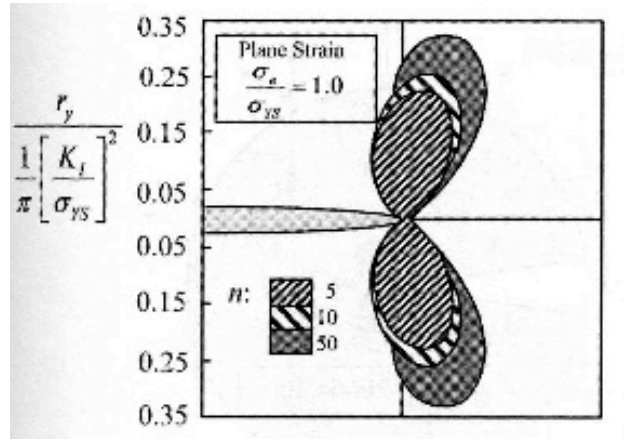
$$\frac{\sigma_1}{\sigma_0} = 2.5$$

When stresses on the crack plane in the linear-elastic field cause incipient yielding, the opening mode stress in the plane-strain model is 2.5 $\times\sigma_0$

Comparison with FE Models

- **Compare "linear-elastic" estimate of plane-strain plastic zone size/shape with results of high-resolution, 2D nonlinear FE analyses**





n **= 5 v. similar to A570 Gr 50**

Uniaxial stress-strain curve: $\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left(\frac{\sigma}{\sigma_0}\right)^n$

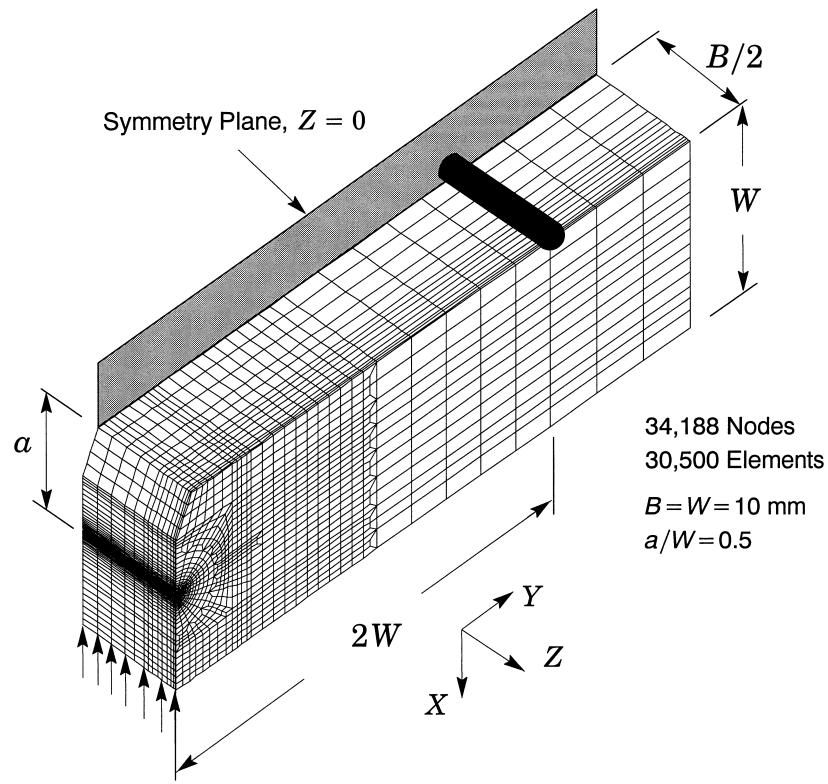
- **An *n* = 50 material is elastic-perfectly plastic. The simple linear-elastic estimation is not bad.**
- **For materials with more strain hardening, the plastic zone is reduced in size as yielded material provides higher stress levels**

3-D effects

Sanford: Chapter 6, pgs. 217-233

Anderson: 2.10

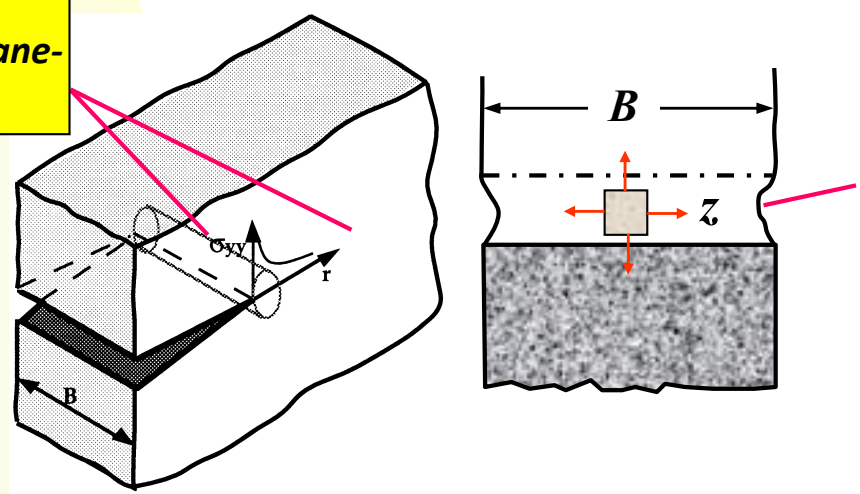
The most up-to-date and technically correct description of the phenomena in a textbook.



3-D effects on crack front

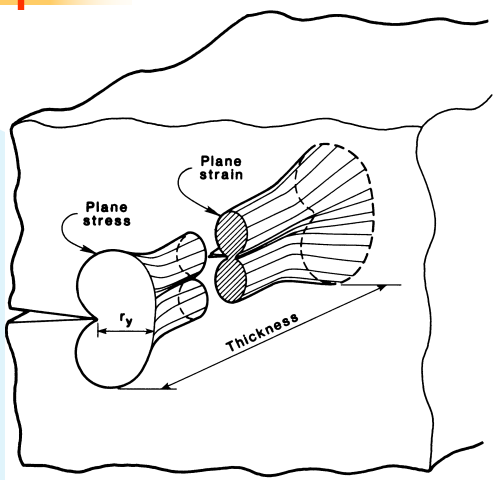
- Closed-form analyses and SSY finite element analyses all assume ***perfect plane-strain or plane-stress conditions***
- FE analyses enable fully 3-D nonlinear solutions and are now quite common
- So-called ***thickness effects*** have lead to much confusion in understanding measured toughness values and development of testing standards, *e.g.*, ASTM E-399

All material outside the immediate crack front region has a simple plane-stress state



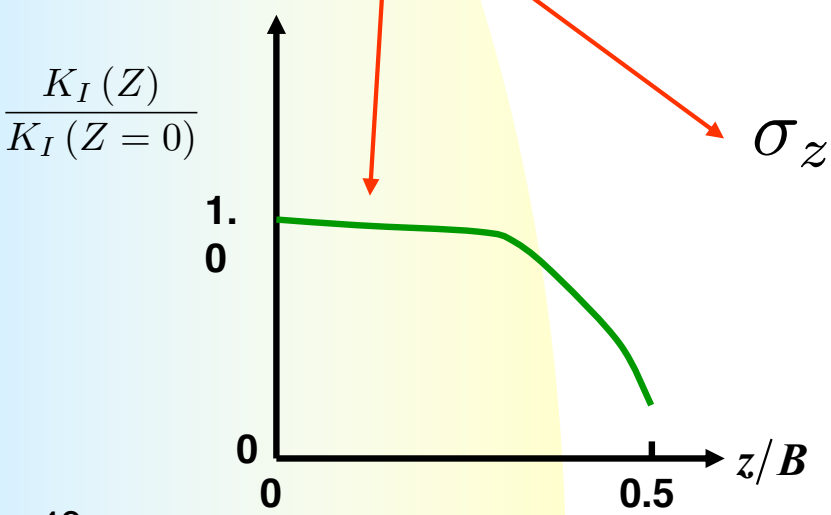
- ***Material very, very near the crack front has large plastic strain***
- ***Wants to contract more than surrounding, less strained material***
- ***Surrounding material "constrains" the thickness contraction***
- ***Generates tensile σ_{zz} stresses***
- ***But these must vanish on external free surfaces***

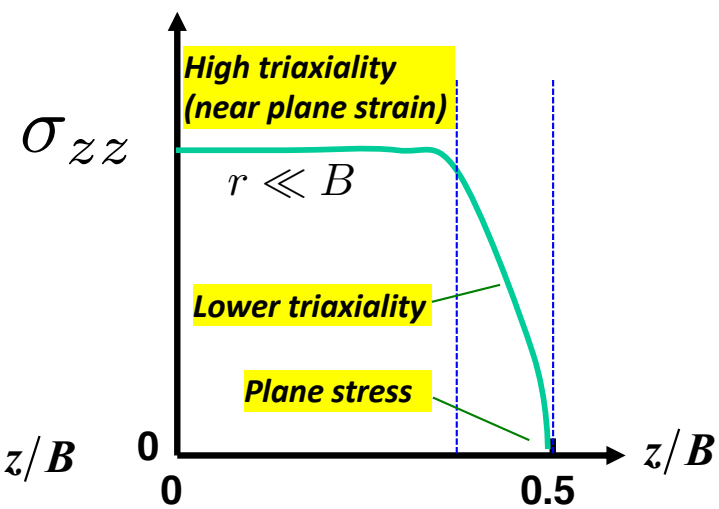
3-D effects on crack front

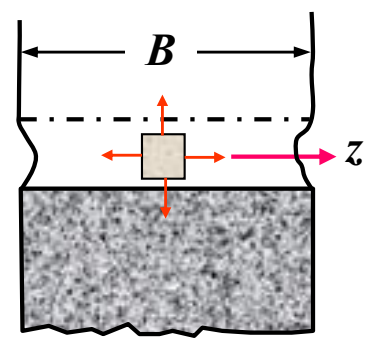


- Schematic drawings like this appear in nearly every fracture mechanics text (pg. 217 – Sanford)
- They show a smaller plastic zone at interior points along the crack front
- Allegedly caused by the "higher constraint" provided by surrounding unyielded material
- These ideas are used to argue for testing of "thick" specimens to achieve "plane-strain" conditions over much of the crack front
- *These figures are not correct !*
- Why? Authors forget that K_I decreases at the outside free surfaces of the specimen
- No singular field exists at the free surface "corners"

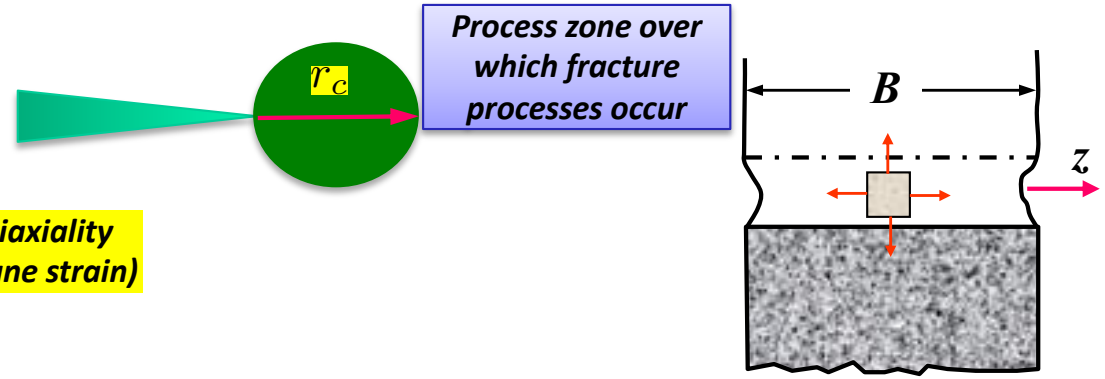
Incorrect for Real Crack Front Because of This

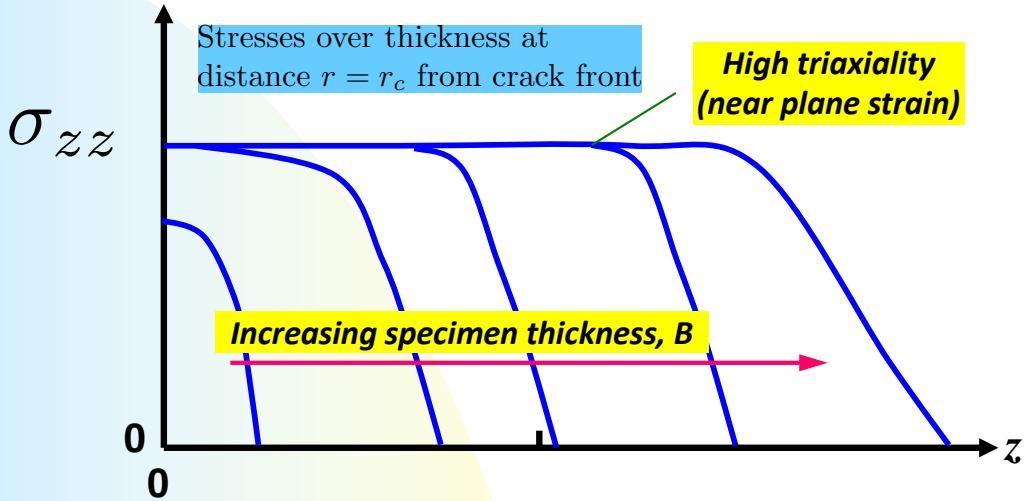


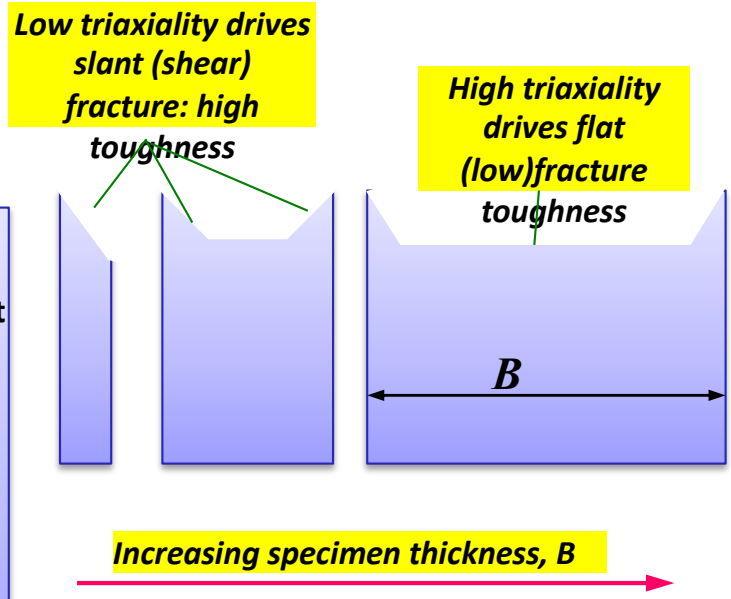




Thickness effects on fields







We observe two fracture mechanisms: flat and slant (also called shear fracture)

For very thin specimens, the centerplane stresses very near the crack front cannot attain the near plane-strain conditions – the outside surfaces are just "too close" and the entire thickness experiences slant fracture (i.e. plane-strain region is smaller than process zone for fracture)

For thicker specimens, the centerplane region near the crack front develops near plane-strain conditions with flat fracture

With further thickness increase, the "relative" portion of flat fracture region increases

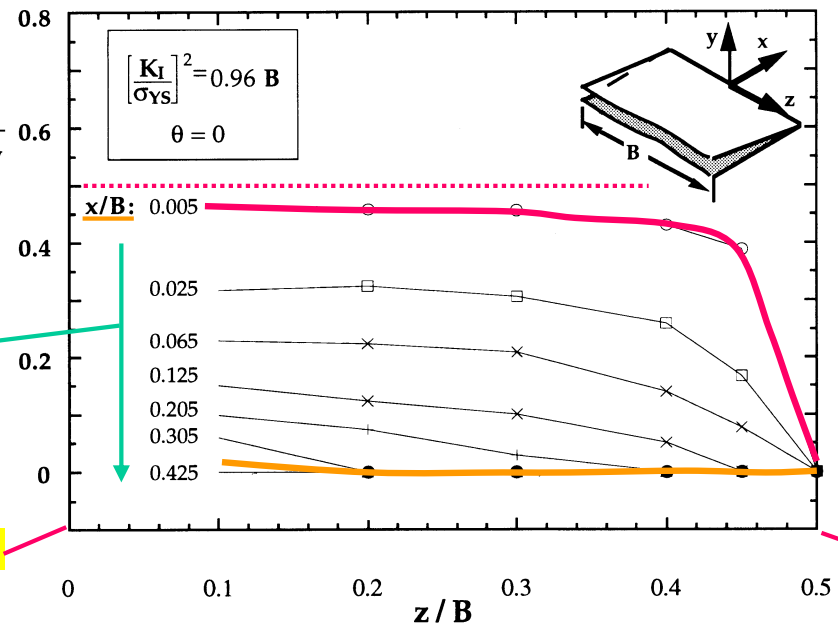
3-D effects on crack front

- 3-D FE analyses demonstrate clearly that surprisingly thin specimens (a few mm) generate opening mode stresses and through thickness stresses near plane-strain levels very near the crack front
- 3-D analyses show that opening mode stresses at mid-plane (*z* = 0) can be maintained to near plane-strain levels to large amounts of plastic deformation
- All stress values decrease near free surfaces at crack front
- At increasing distances from the crack front (*r*) & on the free surfaces at the crack front, the stress-state must be plane-stress

Indicator = 0.5 for perfect plane strain, = 0 for perfect plane stress

Increasing r from crack front at centerplane (x on figure)

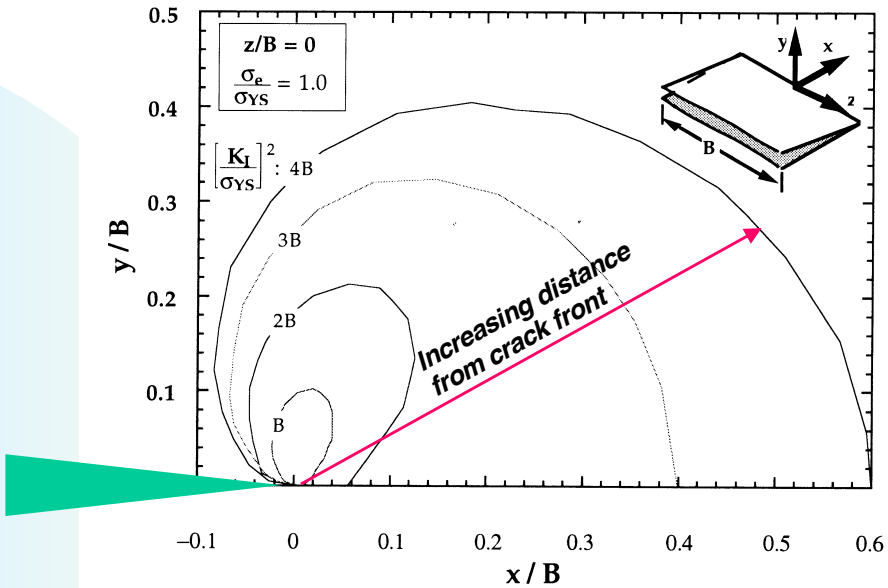
Centerplane (z = 0)



SSY, 3-D Nonlinear Analysis of a Very Large (in-plane) But Thin (B) Panel

Outside surface (z = B/2)

3-D Effects on crack front

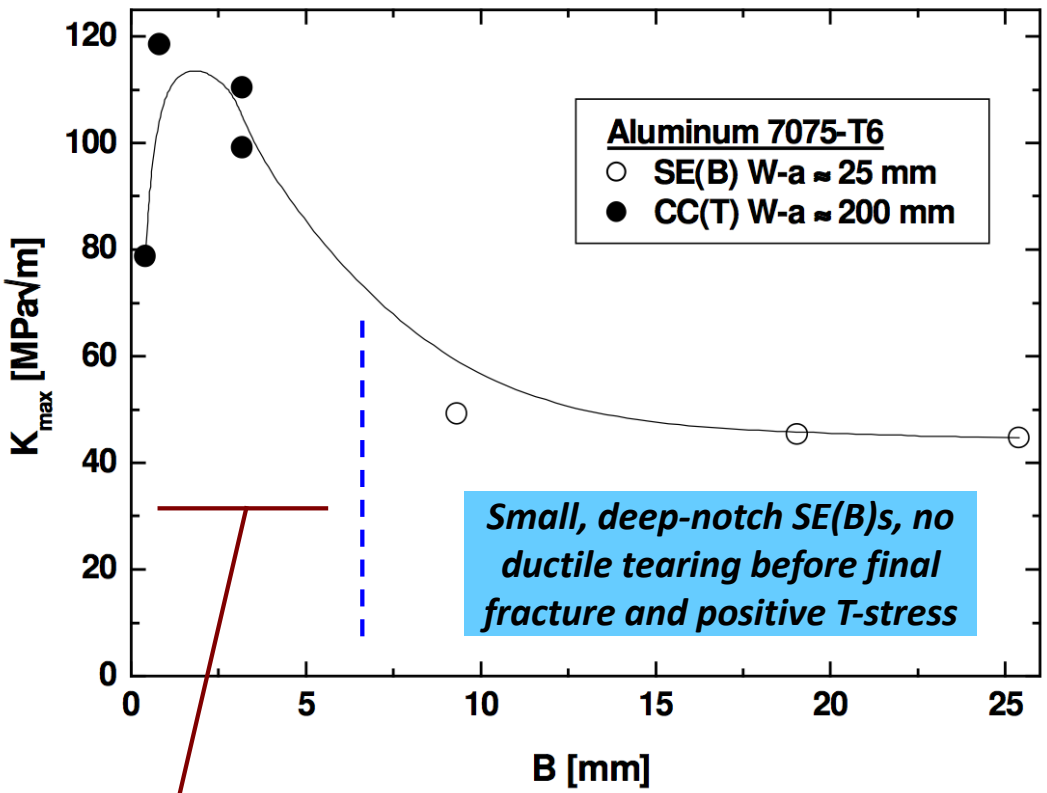


SSY, 3-D Nonlinear Analysis of a Very Large (in-plane) But Thin (B) Panel

Outer boundaries of plastic zone shapes on the centerplane under increasing load

- **Curves are edges of plastic zones on longitudinal centerplane (z = 0)**
- **Each curve is for an increased loading level of applied K_I normalized by yield-stress, *e.g.* when $(K_I/\sigma_{ys})^2 = B$, the plastic zone size is ≈ 0.12 × B**
- **At low loading levels, plastic zones have the classic plane-strain shape**
- **At higher loading levels, plastic zones extend well into the plane-stress region away front crack front**

3-D effects on measured toughness



Small, deep-notch SE(B)s, no ductile tearing before final fracture and positive T-stress

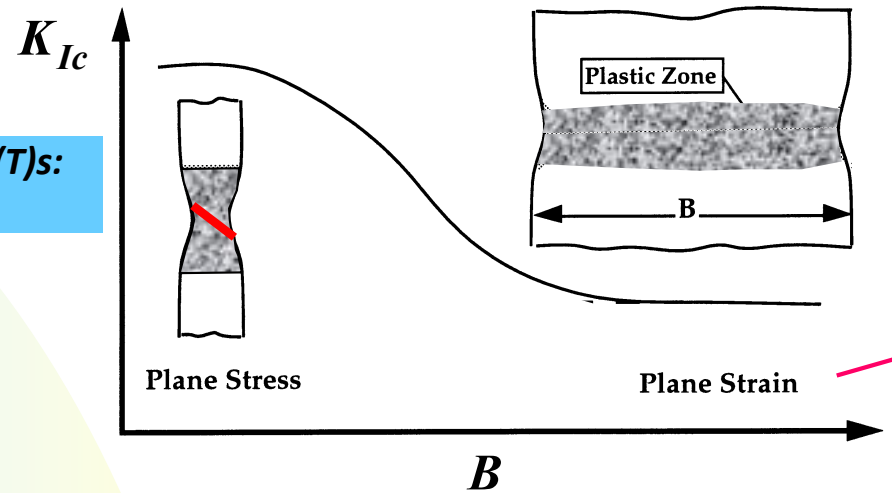
Large M(T)s, lots of ductile tearing before final fracture and negative T-stress

Original data set from 1958 that led to misinterpretations still alive today

Warning!!!!!
This has little to do with "plane strain"

Wallin, K. R. W., "Critical Assessment of the Standard ASTM E 399," *Fatigue and Fracture Mechanics: 34th Volume, ASTM STP 1461*, S. R. Daniewciz, J. C. Newman and K.-H. Schwalbe, Eds., ASTM International, West Conshohocken, PA, 2004.

3-D effects on crack front



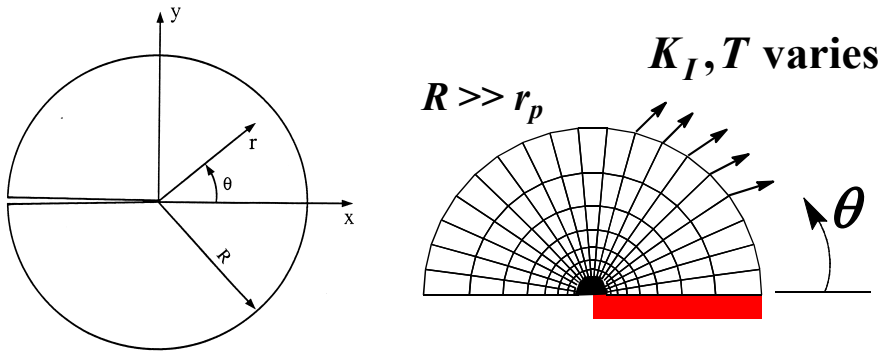
Most often deep-notch SE(B)s and C(T)s: T > 0

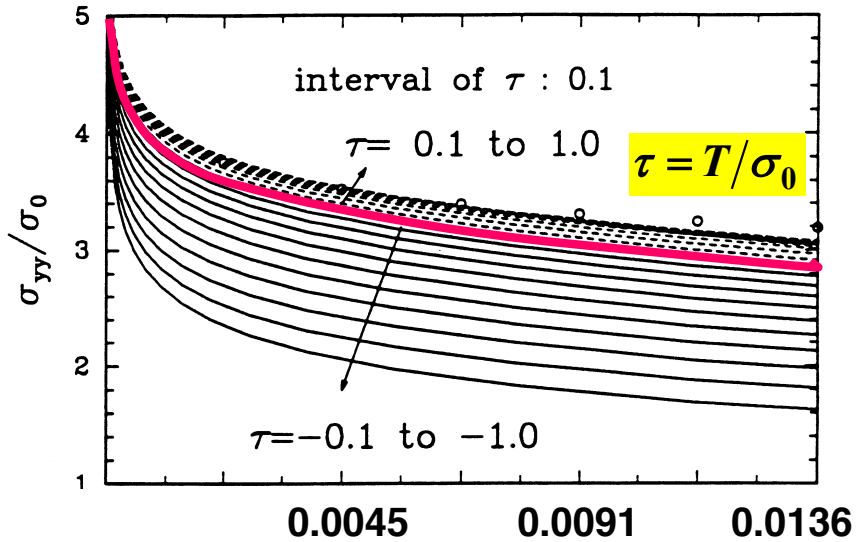
Most often M(T)s: T < 0

Warning!!!!!
This has little to do with "plane strain"

- A "classic plot" to argue thickness effects on fracture toughness
- Confusing because test specimens exhibit a combination of *T*-stress, ductile tearing & thickness (*B*) effects on measured toughness values
- "Old" fracture mechanics never considered *T*-stress (really in-plane effects on toughness) and amounts of ductile tearing [unknown at the time]
- By requiring a "large thickness", "Old" fracture mechanics actually tried to:
  - Insure plastic zone size at fracture is small relative to any in-plane dimension ***and*** thickness
  - Insure that specimen response is really linear-elastic except right at front (SSY)
  - Insure that the crack front has a *T*-stress > 0 (maintains high opening mode stresses)
  - Fracture mode is predominantly flat and with small amounts of prior ductile tearing

T-stress effects on fields





$$r* = \frac{r}{(K_I/\sigma_0)^2} \approx \frac{r}{r_p}$$

SSY, *Plane-Strain*, Mode I, FE Model

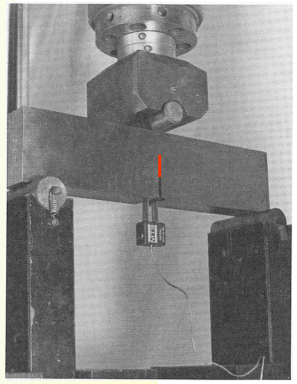
- **How do we know the claims about "thickness" are misleading?**
- **Look at plots of opening mode stresses for *plane-strain* SSY analyses with negative *T*-stress**
- **If we impose plane-strain in the FE analyses, the thickness is ∞**
- **However, with *perfect plane-strain* the stresses fall well below those for *T* = 0**
- **Specimen configurations with *in-plane dimensions* that generate *T* < 0 will measure larger fracture toughness values than *T* > 0 specimens – *independent of thickness !***

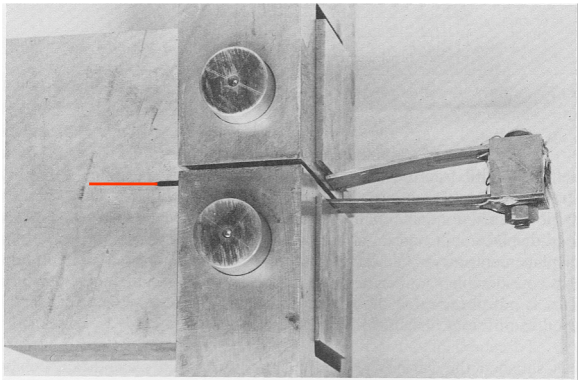
Summary: 3-D effects

- ***Most textbooks are incorrect*** **in their description of 3-D effects on crack front plasticity and the resulting effects on measured fracture toughness**
- **Modern 3-D nonlinear analyses now show this clearly: even thin specimens develop substantial thru-thickness stress**
- **"Old" arguments about needing a large thickness to have "plane strain" really led to large in-plane dimensions of test specimens to maintain SSY conditions**
- **"Thickness" and in-plane size requirements of ASTM E-399 are very conservative for most metals:** ***standards are slowly recognizing the dominant importance of in-plane size and the corresponding effects of T-stress***

K_I as a fracture criterion

Sanford: Chapter 6
Anderson: 2.9, 2.10, Chapter 7.1, 7.2





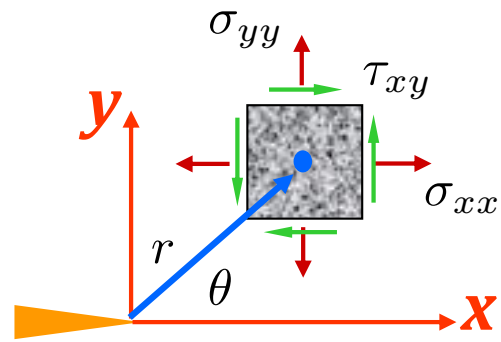
Basis of Correlative Fracture Mechanics

K_I as a fracture criterion

$$\sigma_{yy}(r,\theta) = \frac{K_I}{\sqrt{2\pi\, r}} f_{yy}(\theta)$$

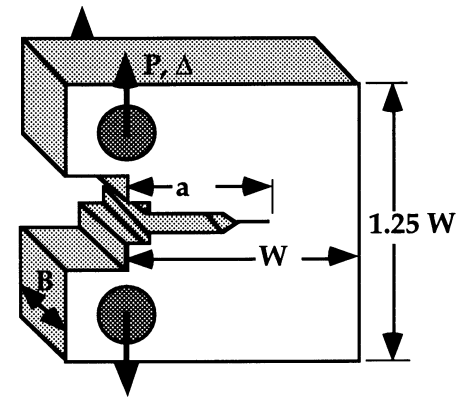
$$\tau_{xy}(r,\theta) = \frac{K_I}{\sqrt{2\pi\, r}} f_{xy}(\theta)$$

$$\sigma_{xx}(r,\theta) = \frac{K_I}{\sqrt{2\pi\, r}} f_{xx}(\theta) + T$$



- **For purely linear-elastic behavior, we know *ALL* information about geometry, crack size, type of loading is contained in the values of K_I and T**
- **If two cracked bodies have the same K_I and T, they have *EXACTLY* the same**
  - **Near tip strains, stresses, and displacements** $r \to 0$
  - **As** $r \to 0$
- **Then, any other quantities computed from the strains and stresses** $(\sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2, \ldots)$ **will also be identical for the same K_I and T as**
  - **Principal values**
  - **Energy densities (U)** $(\sigma_e, \varepsilon_e \ldots)$
  - **Equivalent strains-stresses**

ASTM E-399-05 (2005)



- Requirements for specific specimen types, *e.g.* C(T), and *a/W* ratio limits insure positive *T*-stress geometry
- Specimen "size" requirements often are stated in the form:

$$a, (W-a), \boxed{B} > \varsigma \left(\frac{K_{Ic}}{\sigma_0}\right)^2$$

$\propto r_p$

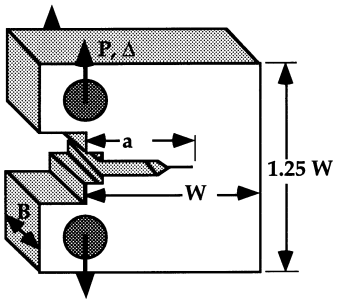
deleted in 2005 version progress !

- E-399 specifies: $\varsigma = 2.5$
  - For an SE(B), $a/W = 0.5$ with $K_{Ic} = 45$ ksi•in$^{1/2}$ and 60 ksi yield stress, the req'd specimen size is $W = 2.8$ in. and $B = 1.4$ in. – *ouch !*
  - The ligament and thickness are > 17 $\times$ r_p
- A more realistic number is $\varsigma = 1.0$ for steels and aluminums to maintain linear-elastic conditions at fracture (based on 3-D FE analyses)
  - For same SE(B), the req'd specimen size is $W = 1.2$ in. and $B = 0.6$ in.
  - The ligament and thickness are > 9 $\times$ r_p

ASTM E-1820-05

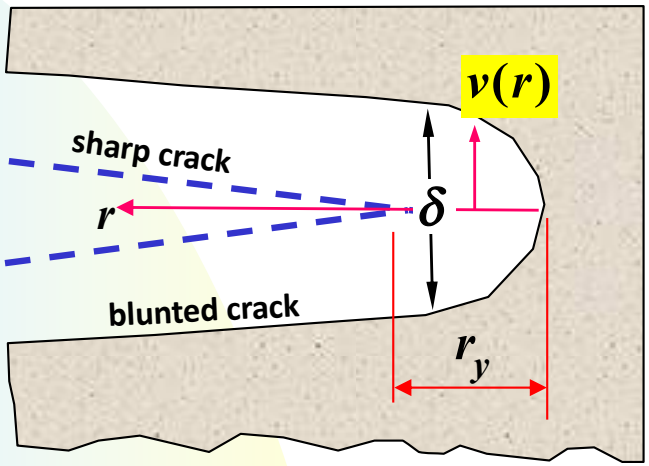
- E-1820 measures an *elastic-plastic* toughness value intended to be under conditions of near SSY
- E-1820 specifies γ = 100 :

$$a, (W-a), B > \gamma \overbrace{\frac{K_{Ic}^2 \left(1-\nu^2\right)}{\sigma_0 E}}^{\propto \delta_c \text{ (see next pg)}}$$



- For a steel SE(B), a/W = 0.5 with K_{Ic}= 100 ksi•in$^{1/2}$ and 60 ksi yield stress, the req'd specimen size is W = 1.0 in. and B = 0.5 in.
  - The ligament and thickness are $\approx r_p$ (plastic hinge allowed @ fracture)
- These much relaxed requirements are based on refined elastic-plastic 3-D analyses of the specimens
- The crack front fields at mid-plane (z = 0) match plane-strain SSY levels over the fracture process zone for metals
- The load-deflection curve for the specimen can be nonlinear at fracture

Reminder: K_I – $CTOD$ relation



$$\delta = \frac{4}{\pi}\frac{K_I^2}{\sigma_0 E}$$

For plane-strain, add the 1-ν^2

Cracked body energetics (1)

- **Total potential energy (π) of a conservative system**

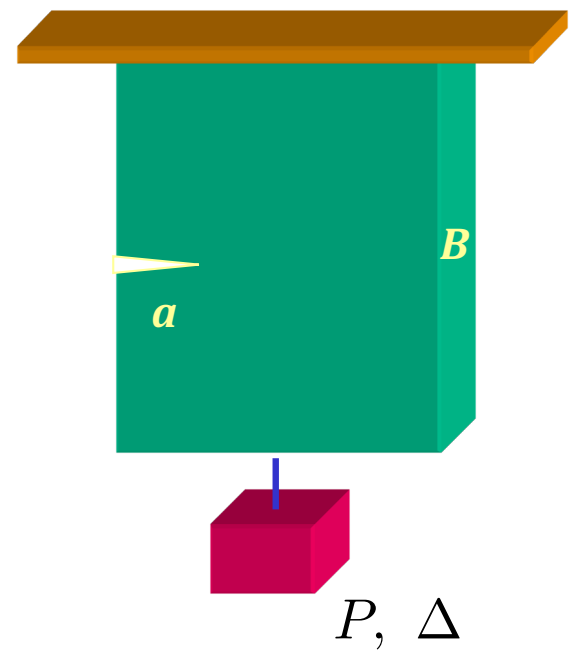
$$\pi = U + \Omega$$

Stored elastic energy

Potential energy of applied loads

$$U = \int P\, d\Delta = \int_V \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij}$$

$$\Omega = -P\Delta = -\iint_{S_T} t_i u_i\, dS = -W_{ext}$$



Load control test

Cracked body energetics (2)

- **If specimen is deformed by imposing displacement (Δ) rather than weight (*P*), then Ω = 0 (*use an old style screw machine in the lab!*)**

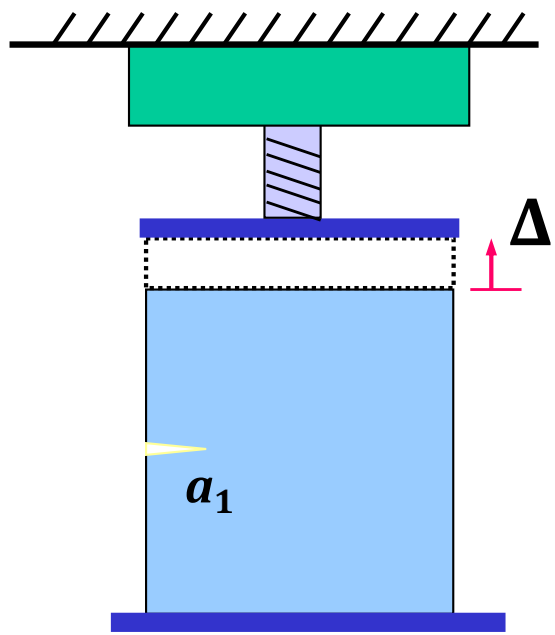
$$\pi = U + \Omega$$

Potential energy of applied loads

Stored elastic energy

$$U = \int P \, d\Delta = \int_V \int_0^{\varepsilon_{ij}} \sigma_{ij} \, d\varepsilon_{ij}$$

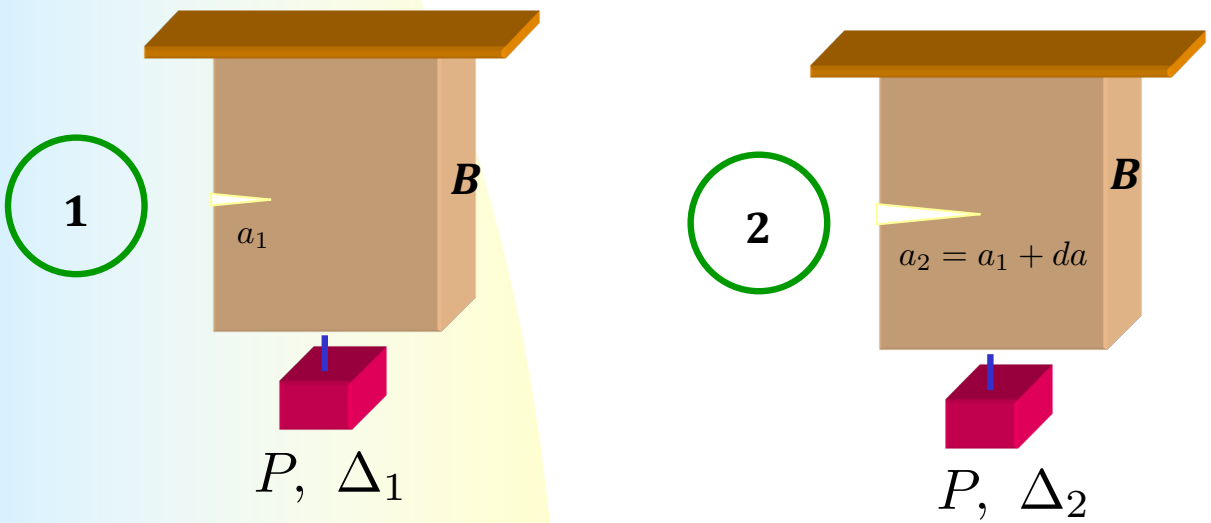
$$\underline{\Omega = 0}$$



Testing machine is so stiff relative to specimen that Δ does not change during crack advance

Energy release rate (1)

- When the crack grows by amount Δa during a test, the new traction free crack area is $\Delta A_c = \Delta a \times B$ (this is conventional usage of the "projected area")
- The crack growth causes the potential energy of the structure to change



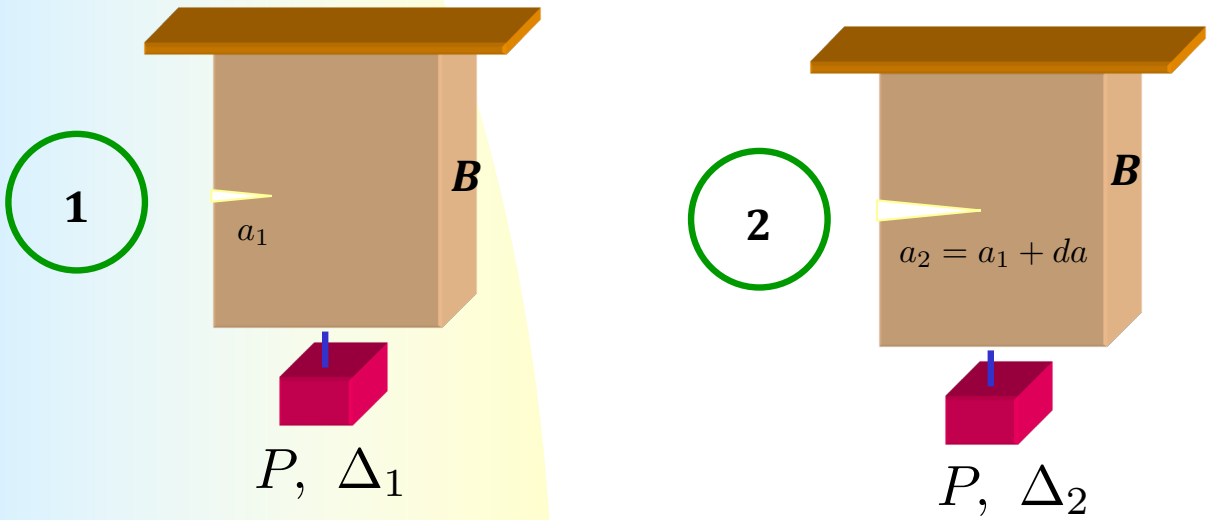
Example shown here is load control during crack extension

Energy release rate (2)

- **The potential energy change per unit area of crack extension is called the *<u>energy release rate</u>* (J or $\mathcal{G}$) and is given by**

$$J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} -\frac{\pi_2 - \pi_1}{B \Delta a}$$

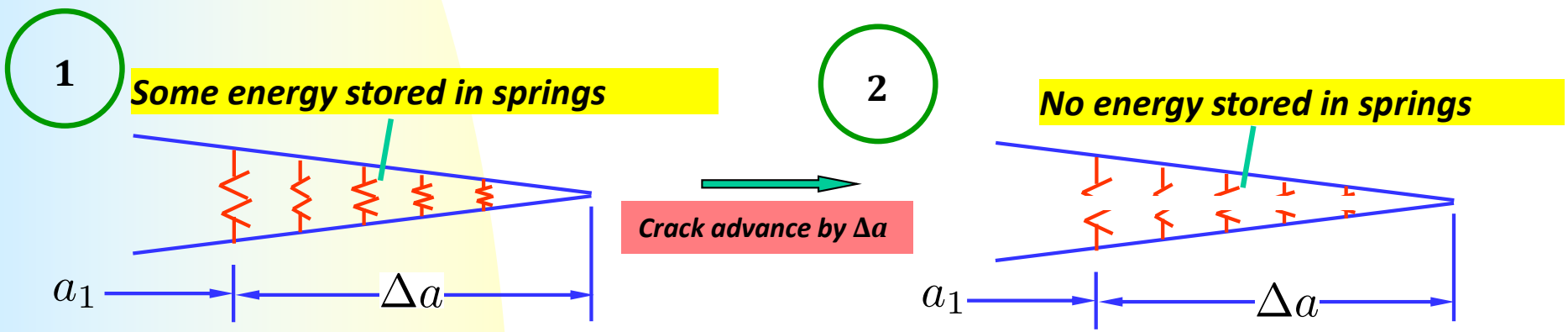
A leading (-) is included in the definition so that the release rate has a positive numerical value



Example shown here is <u>load control</u> during crack extension

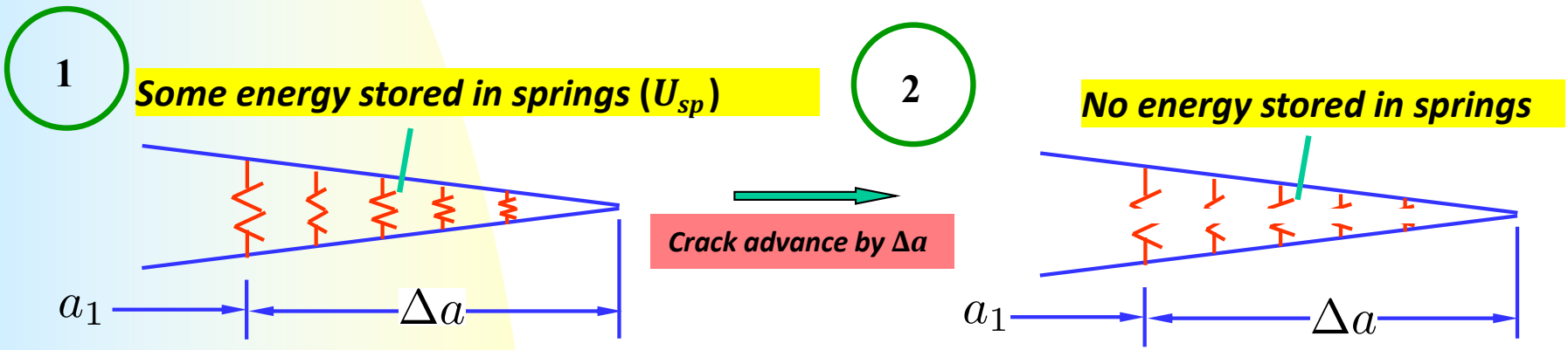
J for Displacement Controlled Loading

- The key concept is what happens with the forces (tractions) across the crack area $\Delta a \times B$ in configuration 1 that are zero in configuration 2 after the crack grows
- Imagine that we have simple springs connecting the crack faces over $\Delta a \times B$
- In 1, these springs are stretched due to specimen loading and they have some stored elastic energy
- In 2, these springs are broken (to create crack growth by $\Delta a \times B$) and no longer have the stored energy

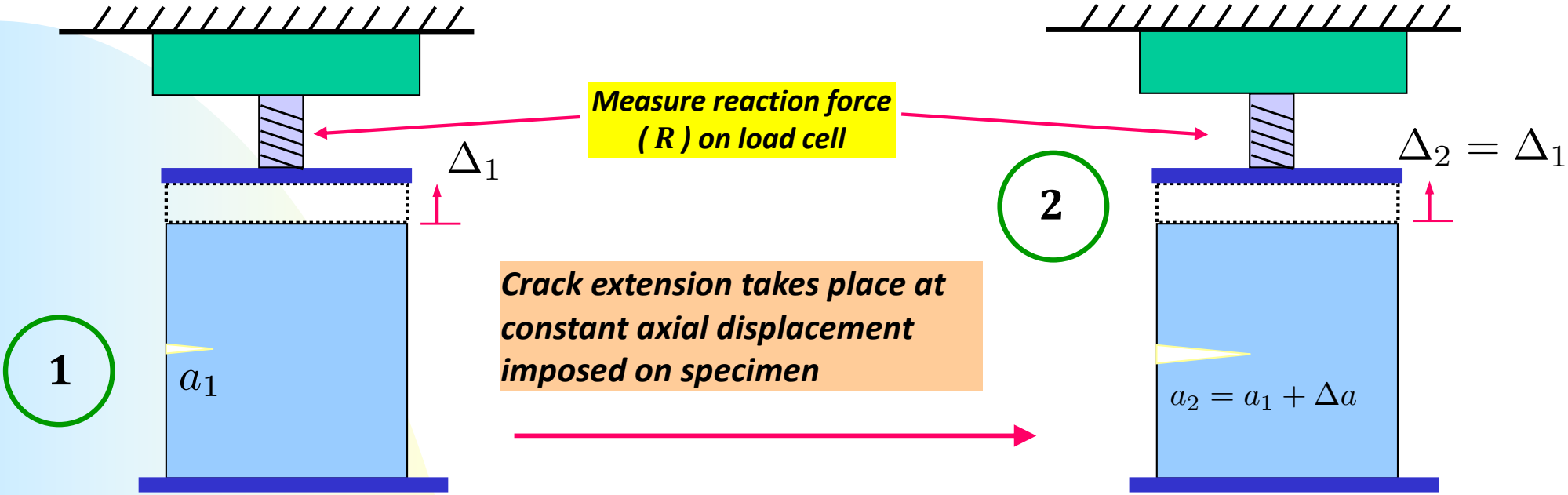


J for Displacement Controlled Loading

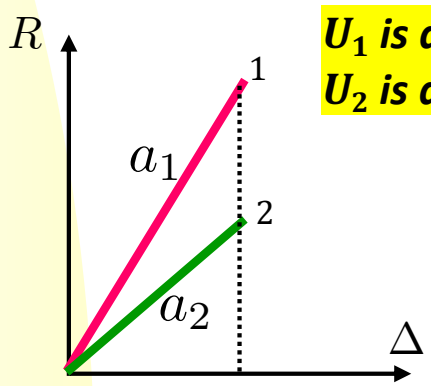
- Something happens to the energy balance in the cracked structure since it is <u>closed</u> and <u>conservative</u> – this changes the potential energy
- We also know that for fixed displacement loading, $\Omega = 0$ in 1 and in 2, and that the *compliance (flexibility) increases* in 2 due to the increased crack length



Compute J : Δ controlled loading (1)



$$\pi_1 = U_1$$
$$\Omega_1 = 0$$



U_1 is area under red line
U_2 is area under green line

$$\pi_2 = U_2$$
$$\Omega_2 = 0$$

Specimen compliance is inverse of stiffness.
Compliance increases during crack advance.

Compute J : Δ controlled loading (2)

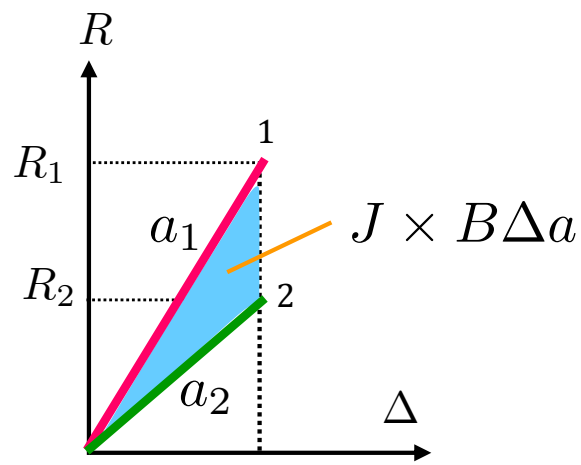
U_1 is area under red line and includes U_{sp}
U_2 is area under green line: $U_{sp} = 0$

$$\pi_2 - \pi_1 = U_2 - U_1$$

$$J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} \; -\frac{\pi_2 - \pi_1}{B \Delta a} = -\frac{U_2 - U_1}{B \Delta a}$$

$$U_1 = \frac{1}{2} R_1 \, \Delta; \quad U_2 = \frac{1}{2} R_2 \, \Delta$$

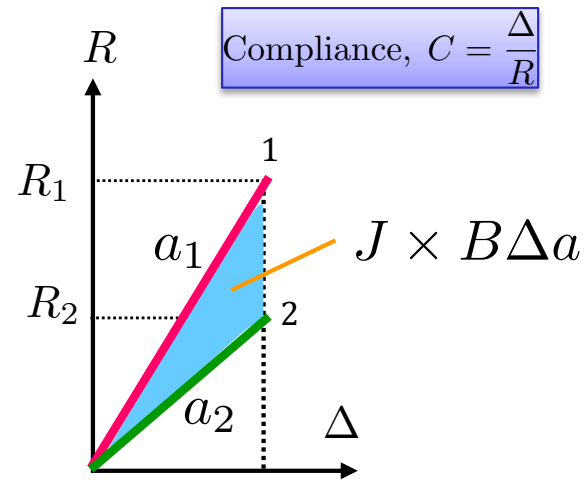
$$J = \lim_{\Delta a \to 0} \; -\frac{\Delta}{2B} \frac{R_2 - R_1}{\Delta a} = -\frac{\Delta}{2B} \left(\frac{\partial R}{\partial a} \right)_{\Delta = fixed}$$



J has units of F-L / L²

Compute J : Δ controlled loading (3)

- **Write R and Δ in terms of compliance C: C = Δ / R**
- **Then the derivative can be re-written to give**



$$J = -\frac{\Delta}{2B}\left(\frac{\partial R}{\partial a}\right)_{\Delta=fixed} = -\frac{\Delta}{2B}\underbrace{\left(\frac{\partial\left(\Delta/C\right)}{\partial a}\right)_{\Delta=fixed}}$$

= 0 *(fixed grips, Δ does not change with a)*

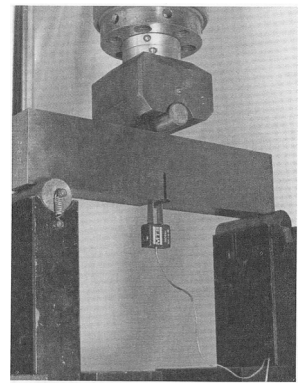
$$\left(\frac{\partial\left(\Delta/C\right)}{\partial a}\right)_{\Delta=fixed} = \frac{C\left(\cancel{\frac{\partial\Delta}{\partial a}}\right)_{\Delta=fixed} - \Delta\left(\frac{\partial C}{\partial a}\right)_{\Delta=fixed}}{C^2}$$

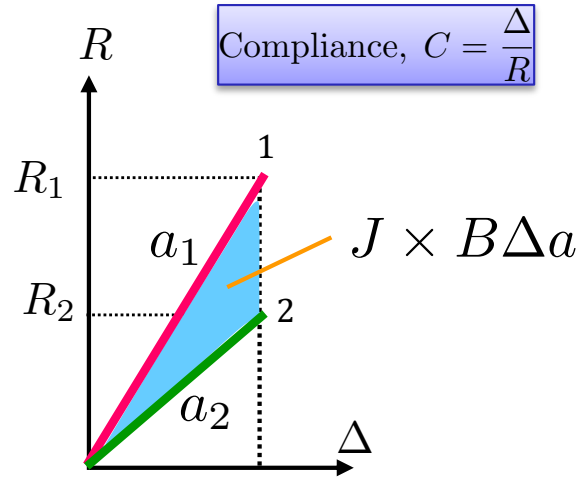
$$= -\frac{\Delta}{C^2}\left(\frac{\partial C}{\partial a}\right)_{\Delta=fixed}$$

$$J = -\frac{\Delta}{2B}\left(-\frac{\Delta}{C^2}\right)\left(\frac{\partial C}{\partial a}\right)_{\Delta=fixed} = \frac{1}{2B}\frac{\Delta}{C}\frac{\Delta}{C}\left(\frac{\partial C}{\partial a}\right)_{\Delta=fixed} = \frac{R^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{\Delta=fixed}$$

Compute J : Δ controlled loading (3)

$$J = \frac{R^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{\Delta = fixed}$$



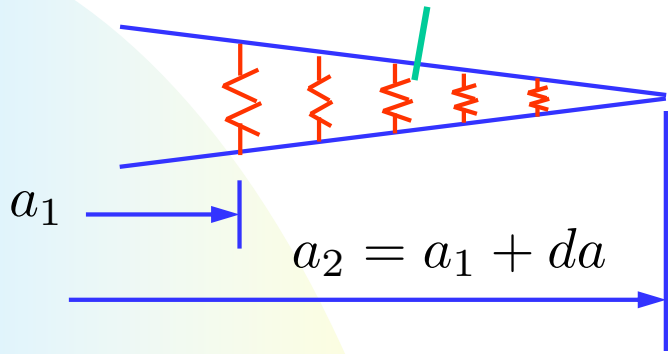


How to Use ?

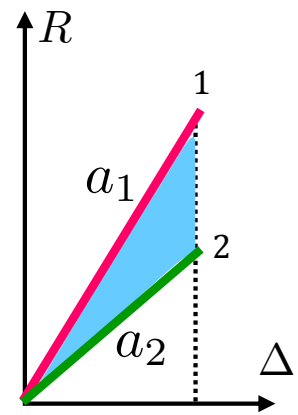
- **Suppose we have a closed-form solution for the compliance**
- **Energy release rate follows directly by applying the above derivative (usually v. simple to compute)**
- **Can also compute <u>approximate compliance change</u> in a finite element analysis for small changes in crack length**

J interpretation

U_{sp} is energy stored in springs



U_1 is area under red line: includes U_{sp}
U_2 is area under green line: U_{sp}= 0



- **Since the system is closed, conservative, & fixed displacement during growth, the energy change between 1 and 2 must be *only* from the energy stored in the broken springs**

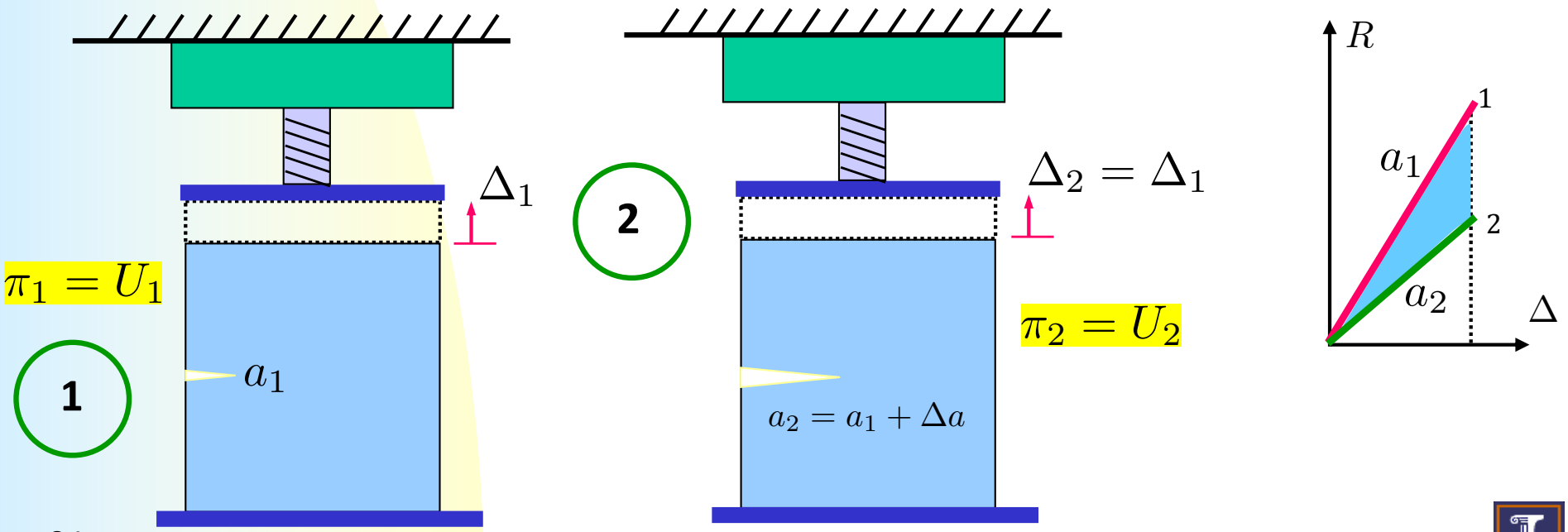
$$\left.\begin{aligned} U_2 + U_{sp} &= U_1 \\ U_2 - U_1 &= -U_{sp} \end{aligned}\right\} \rightarrow \pi_2 - \pi_1 = U_2 - U_1 = -U_{sp}$$

$$J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} -\frac{\pi_2 - \pi_1}{B\Delta a} = -\frac{-U_{sp}}{B\Delta a} = \frac{U_{sp}}{dA_c}$$

J has units of F-L / L^2

What does this mean?

- When K_I approaches K_{Ic}, the cracked body in configuration 1 has just reached sufficient potential energy (in this case, stored elastic energy) available (and can give up) to break the springs (cohesive tractions) holding the crack closed over the area $B \times \Delta a$ (*i.e.* the energy to be ***released*** during crack advance)
- When K_I from the applied displacement is < K_{Ic}:
  - The cohesive tractions (*i.e.* spring forces or strength) ***available*** from the metallurgical features exceed the applied tractions imposed by K_I stress field (no break)
  - The energy available from the background (elastic) material during a crack growth $B \times \Delta a$ < energy stored in the springs at their breaking point



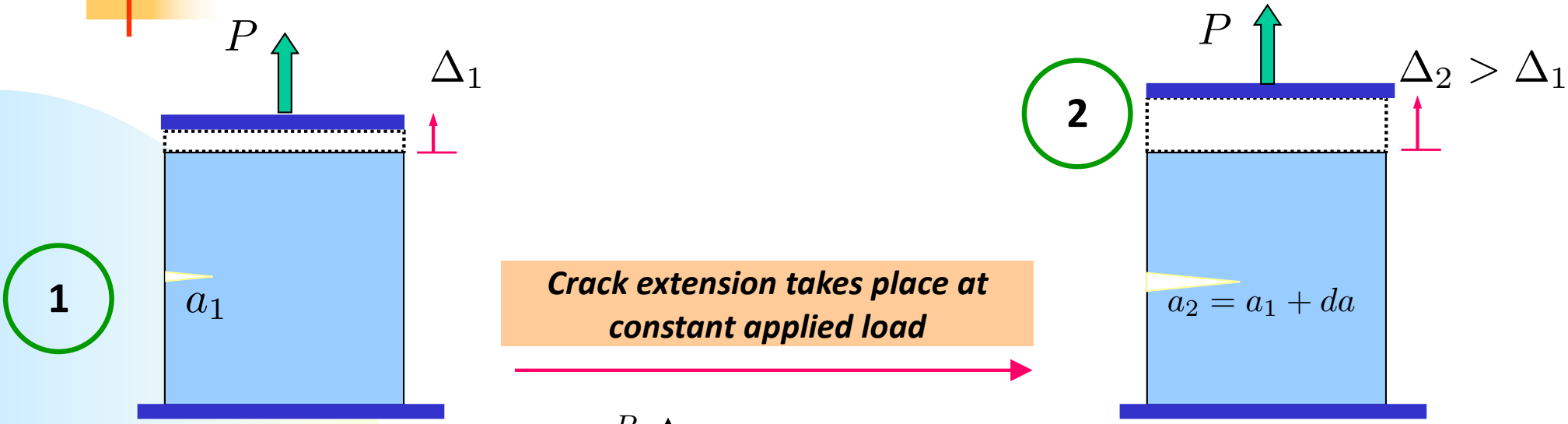
What does this mean? (2)

- The <u>energy</u> criterion was the original concept of fracture mechanics: $K_I = K_{Ic}$ came many years later
- In a few slides we show that energy and stress-intensity arguments are <u>equivalent for linear-elastic systems</u>
- Energy arguments are especially powerful in finite element analysis: U and Ω are the most accurate quantities computed
- ***<u>Key point:</u>*** fracture conditions described w/o any mention of the metallurgical mechanisms! (caused much confusion for decades – still does)

$$J_{applied} = J_{critical}$$

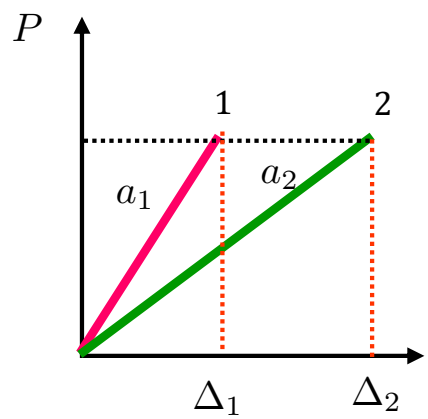
Original concept of fracture mechanics

J for load control (1)



Crack extension takes place at constant applied load

$$\pi_1 = U_1 + \Omega_1$$
$$\Omega_1 = -P\Delta_1$$
$$U_1 = \frac{1}{2}P\Delta_1$$



$$\pi_2 = U_2 + \Omega_2$$
$$\Omega_2 = -P\Delta_2$$
$$U_2 = \frac{1}{2}P\Delta_2$$

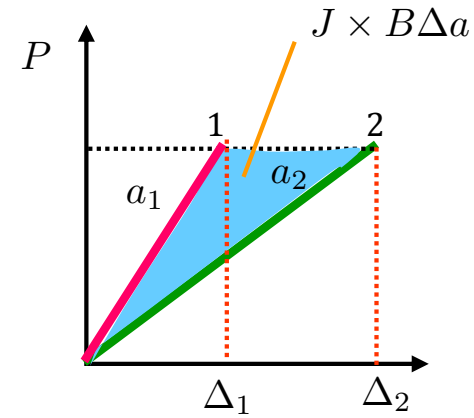
U_1 is area under red line
U_2 is area under green line

J for load control (2)

$$\text{Compliance, } C = \frac{\Delta}{P}$$

$$\left.\begin{aligned}\Omega_1 &= -P\Delta_1 \\ U_1 &= \frac{1}{2}P\Delta_1\end{aligned}\right\} \quad \pi_1 = U_1 + \Omega_1 = -\frac{1}{2}P\Delta_1$$

$$\left.\begin{aligned}\Omega_2 &= -P\Delta_2 \\ U_2 &= \frac{1}{2}P\Delta_2\end{aligned}\right\} \quad \pi_2 = U_2 + \Omega_2 = -\frac{1}{2}P\Delta_2$$

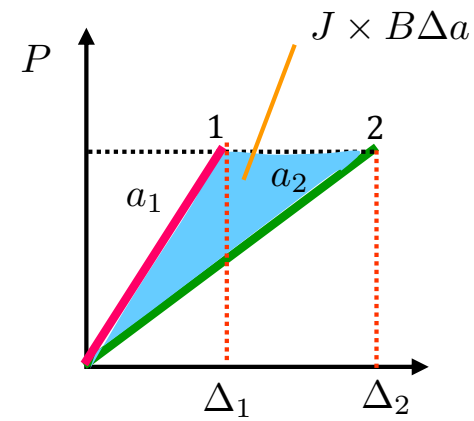


$$J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} -\frac{\pi_2 - \pi_1}{B\Delta a} = \frac{1}{2}\frac{P\left(\Delta_2 - \Delta_1\right)}{B\,da} = \frac{P}{2B}\left(\frac{\partial \Delta}{\partial a}\right)_{P=fixed}$$

J for load control (3)

$$J = -\frac{\partial \pi}{\partial A_c} = \frac{P}{2B}\left(\frac{\partial \Delta}{\partial a}\right)_{P=fixed}$$

Compliance, $C = \dfrac{\Delta}{P}$



- **Write *P* and Δ in terms of compliance *C*: *C* = Δ / *P***
- **Then the derivative can be re-written to give**

$$J = \frac{P}{2B}\left(\frac{\partial \Delta}{\partial a}\right)_{P=fixed} = \frac{P}{2B}\underbrace{\left(\frac{\partial (PC)}{\partial a}\right)}_{}_{P=fixed}$$

= 0 *(fixed load, P does not change with a)*

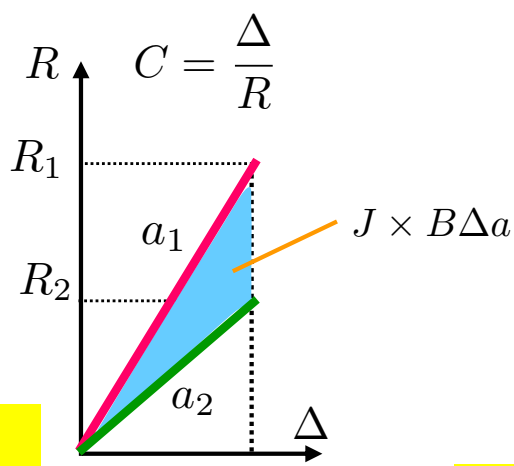
$$\left(\frac{\partial (PC)}{\partial a}\right)_{P=fixed} = C\overset{0}{\cancel{\left(\frac{\partial P}{\partial a}\right)_{P=fixed}}} + P\left(\frac{\partial C}{\partial a}\right)_{P=fixed}$$

$$= P\left(\frac{\partial C}{\partial a}\right)_{P=fixed}$$

$$J = \frac{P^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{P=fixed}$$

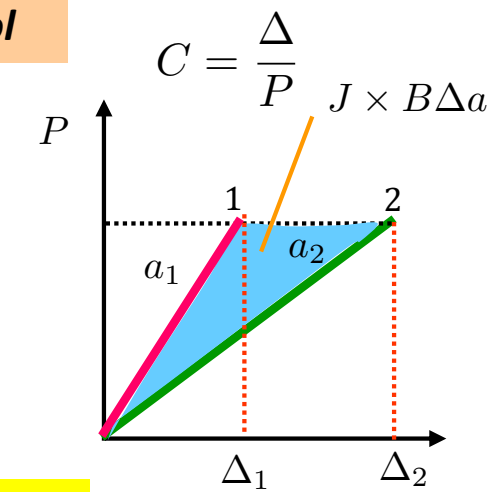
Equivalence of load-displacement control (1)

Displacement Control



$$J = \frac{R^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{\Delta = fixed}$$

Load Control

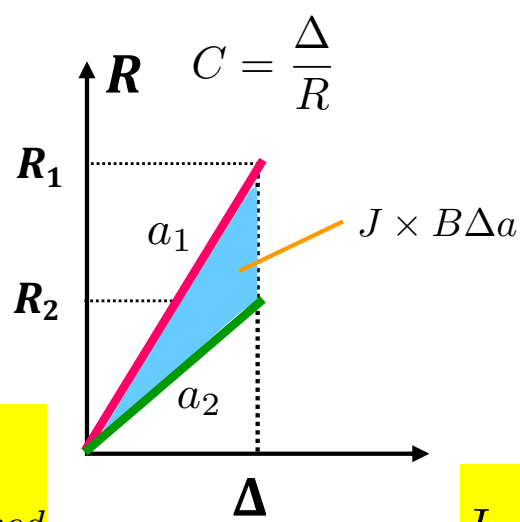


$$J = \frac{P^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{P = fixed}$$

- **The crack grows when the reaction from the imposed displacement is the same as the load applied by a weight, *i.e.*, $R = P$ (and the internal stresses are identical)**
- **The compliance, C, of a linear-elastic specimen is not a function of the load P or the displacement Δ. It is a function only of the crack length (and other dimensions of the specimen) end the material elastic properties**

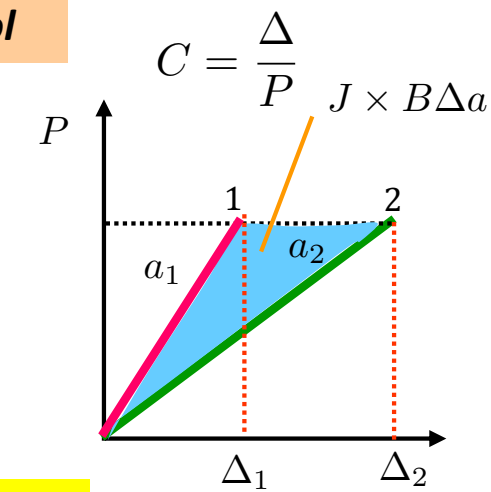
Equivalence of load-displacement control (2)

Displacement Control



$$J = \frac{R^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{\Delta = fixed}$$

Load Control



$$J = \frac{P^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{P = fixed}$$

- **Consequently,**

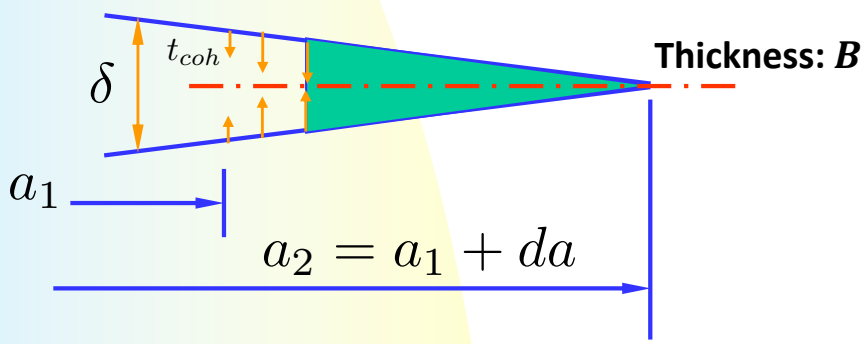
$$R = P$$

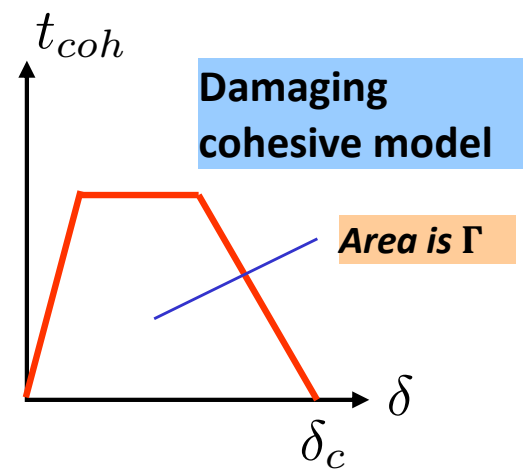
$$\left(\frac{\partial C}{\partial a}\right)_{\Delta = fixed} = \left(\frac{\partial C}{\partial a}\right)_{P = fixed}$$

✓ ***J*** **for Load & Displacement Control are Identical**

General cohesive model (1)

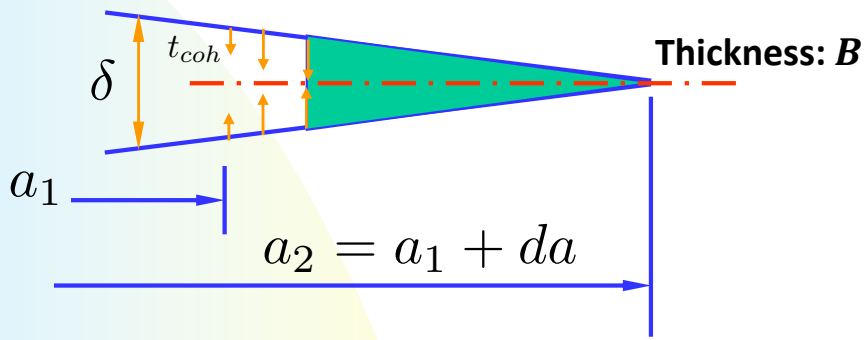
- More realistic models treat the "springs" as <u>cohesive elements</u> uniformly distributed over the crack plane
- The cohesive stress (t_{coh}) normal to the crack plane varies with the opening displacement (δ) between the crack faces
- The cohesive law can be linear, nonlinear or damaging

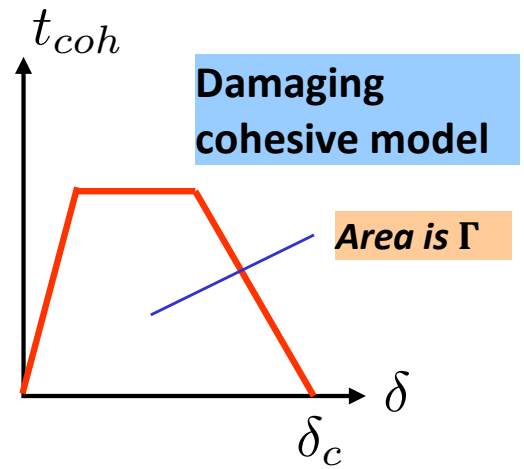




General cohesive model (2)

- **Compute the energy stored in cohesive forces (i.e., U_{sp})**





This calculation is for traction across the crack area & the total displacement across the faces.

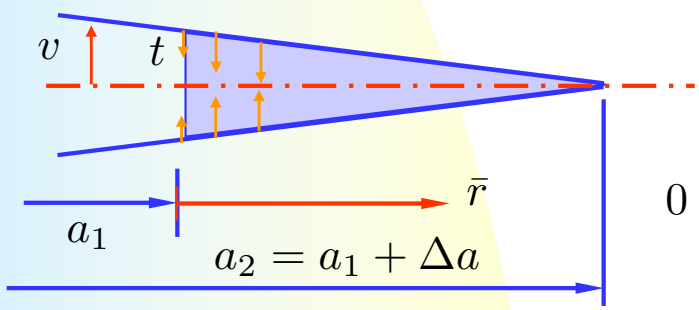
$$U_{sp} = U_{coh} = \int_0^{\delta_c} \underbrace{[t(\delta)\ Bda]}_{\text{traction x area = force}} d\delta = Bda \int_0^{\delta_c} t(\delta)\ d\delta$$

$$J = \frac{U_{coh}}{Bda} = \int_0^{\delta_c} t(\delta)\, d\delta = \Gamma$$

Energy release per new crack face area

J-K Relationship (1)

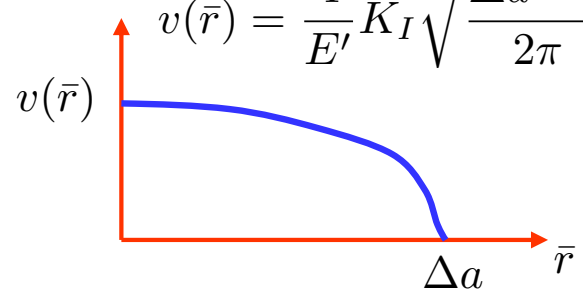
- **Consider Mode I plane-strain conditions**
  - **Opening mode stresses acting ahead of crack in configuration 1 ($a = a_1$) are given by K_I**
  - **Let crack area grow by $B \times \Delta a$ to configuration 2**
  - **The crack stresses *relax* to zero over new crack area $B \times \Delta a$ gradually as the crack opens from a_1 to a_2**
  - **The opening displacements must follow the Mode I solution**
  - **Perform "cohesive" integration to find *J* (work done by the tractions as they relax to zero during displacement increase)**



$$0 \leqslant \bar{r} \leqslant \Delta a$$

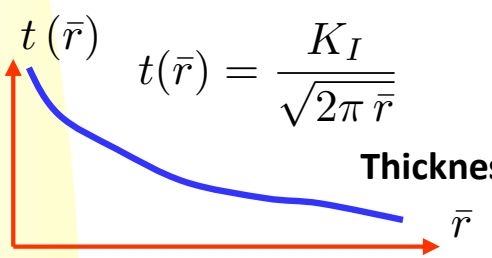
$$r = \Delta a - \bar{r}$$

$$v(\bar{r}) = \frac{4}{E'} K_I \sqrt{\frac{\Delta a - \bar{r}}{2\pi}}$$



$$E' = \frac{E}{1 - \nu^2}$$

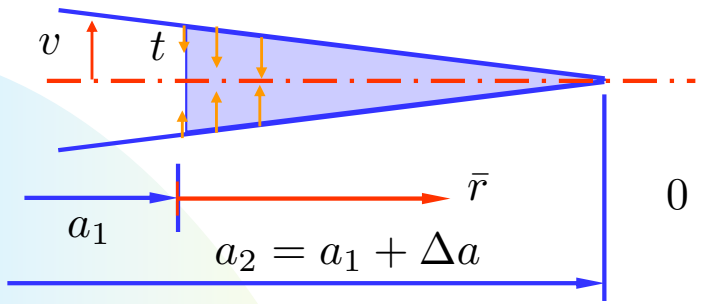
Plane-strain



$$t(\bar{r}) = \frac{K_I}{\sqrt{2\pi\,\bar{r}}}$$

Thickness: *B*

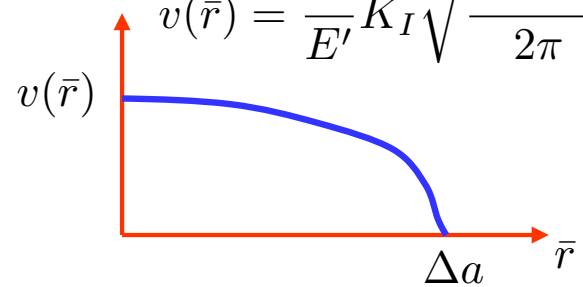
J-K Relationship (2)



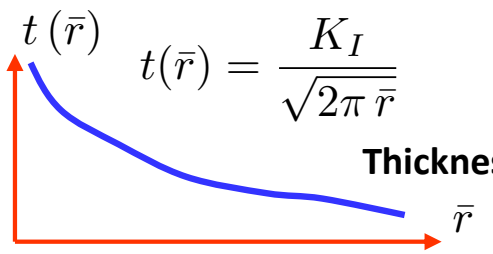
$$0 \leqslant \bar{r} \leqslant \Delta a$$

$$v(\bar{r}) = \frac{4}{E'} K_I \sqrt{\frac{\Delta a - \bar{r}}{2\pi}}$$

$$r = \Delta a - \bar{r}$$



$$t(\bar{r}) = \frac{K_I}{\sqrt{2\pi\,\bar{r}}}$$



Thickness: *B*

$$U_{coh} = 2 \int_0^{\bar{r}=\Delta a} \frac{1}{2} v(\bar{r})\, \underbrace{t(\bar{r})\; B\, d\bar{r}}_{\text{Differential force}} = \frac{4K_I^2}{2\pi E'} B \underbrace{\int_0^{\Delta a} \sqrt{\frac{\Delta a - \bar{r}}{\bar{r}}}\, d\bar{r}}_{\frac{\Delta a \pi}{2}}$$

$$J = \frac{U_{coh}}{B\Delta a} = \frac{K_I^2}{E'}$$

$E' = E$ **Plane-stress**

$E' = \dfrac{E}{1-\nu^2}$ **Plane-strain**

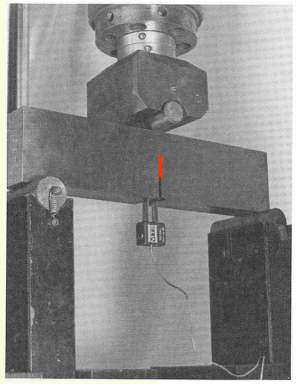
Remember: *these tractions and displacements are the final values (thus a ½ is needed for energy)*
The outside 2 is needed because the displacement (v) is just the upper & lower value about symmetry plane

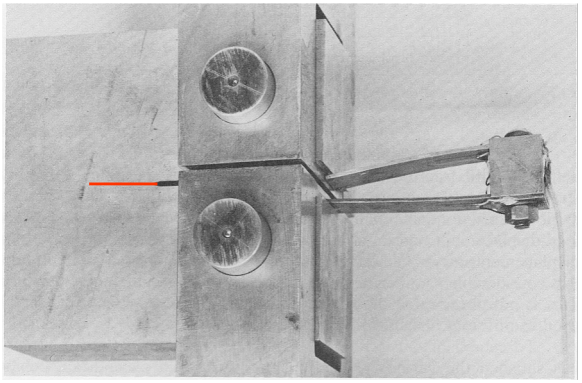
J-K Relationship (3)

- The ***stress-intensity factor*** approach and the ***energy-based*** approach for fracture under linear-elastic conditions are identical!!
- Given K_I we can compute J
- Given J we can compute K_I
- Use the more convenient computational approach for the problem needing solution
- Energy release rates (J) are very accurate even for relatively crude finite element models that employ <u>displacement-based</u> element formulations

Applications of Energy Approach

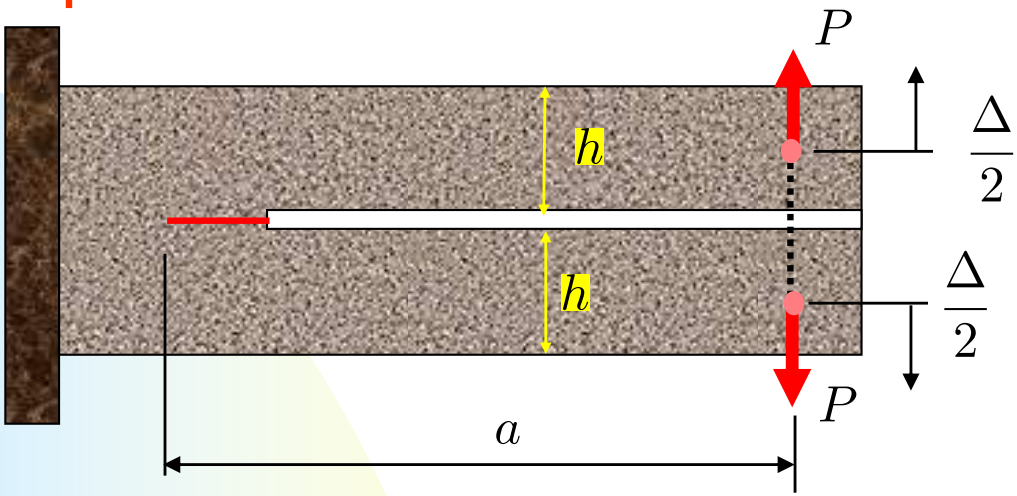
Sanford: Chapter 6
Anderson: 2.3,4,7,10
Kanninen and Popelar: pg. 32-37, 158-163





Equivalence of Stress Intensity and Energy Methods

Example *J* computation (1)



Double Cantilever Beam

Thickness: B

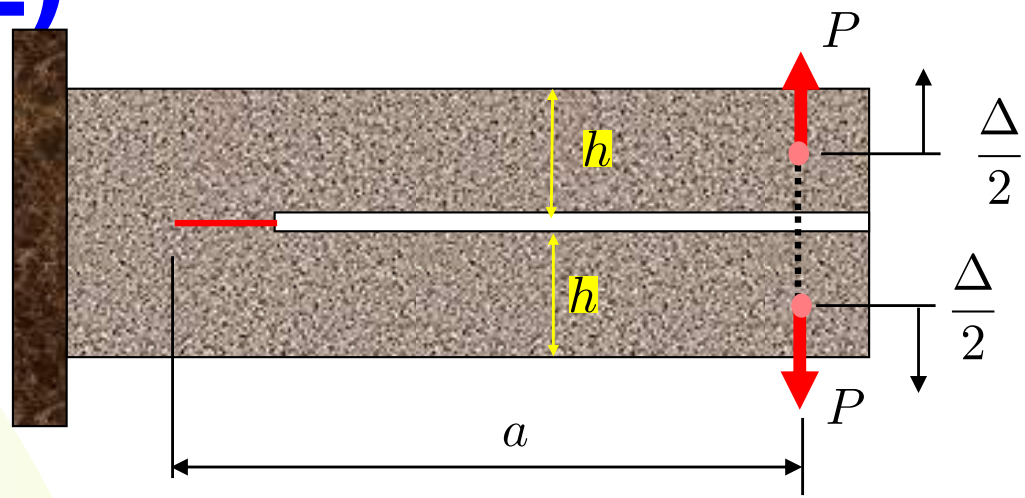
Use simple beam theory to compute the displacement in the arms of the specimen assuming cantilever behavior

$$\frac{\Delta}{2} = \frac{PL^3}{3EI} = \frac{Pa^3}{3EI}; \quad I = \frac{1}{12}Bh^3$$

total displacement

$$Compliance, \quad C = \frac{\Delta}{P} = \frac{2a^3}{3EI}$$

Example *J* computation (2)



Thickness: B

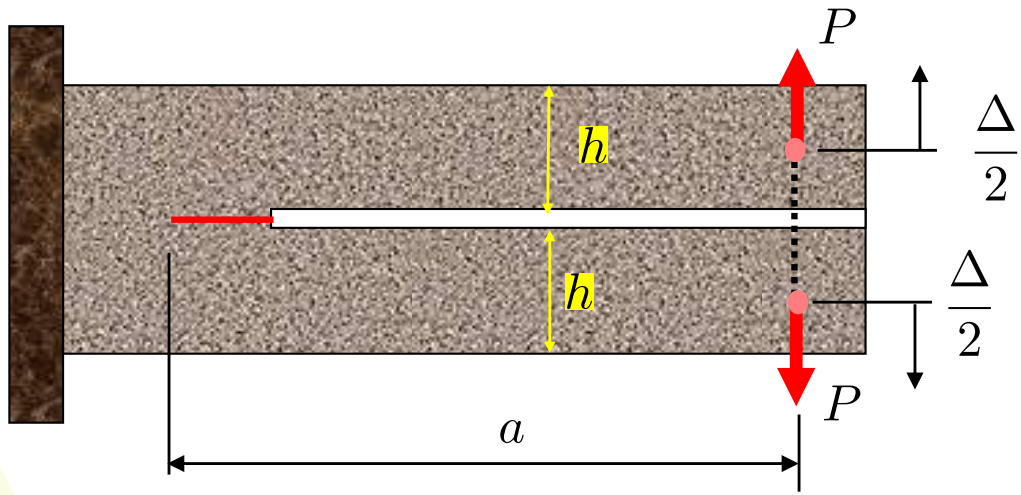
$$\frac{\Delta}{2} = \frac{PL^3}{3EI} = \frac{Pa^3}{3EI}; \quad I = \frac{1}{12}Bh^3$$

$$Compliance, \quad C = \frac{\Delta}{P} = \frac{2a^3}{3EI}$$

$$J = \frac{P^2}{2B}\frac{\partial C}{\partial a} = \frac{P^2}{2B}\frac{\partial}{\partial a}\left(\frac{2a^3}{3EI}\right) = \frac{P^2a^2}{BEI} = \frac{12P^2a^2}{B^2h^3E}$$

$$\boxed{J = \frac{12P^2a^2}{B^2h^3E}}$$

Example *J* computation (3)



Thickness: B

$$J = \frac{12P^2a^2}{B^2h^3E}$$

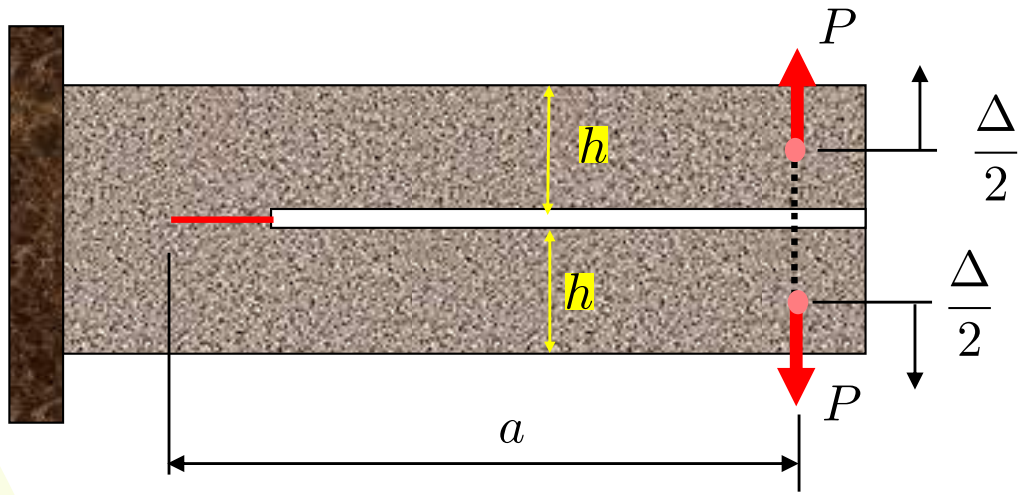
- **Neglects shear deformation in the arms**
- **Assumes arms fixed at crack tip**
- **Neglects energy in remainder of specimen**
- **At constant load, *J* increases as the square of the crack length**
- **Verify correct physical units (must be *F-L/L*²)**

$$K_I = \sqrt{EJ} = \frac{2\sqrt{3}Pa}{Bh^{3/2}}$$

for plane stress

This is a very good approximation, especially for long, slender arms

Crack stability (1)



Thickness: B

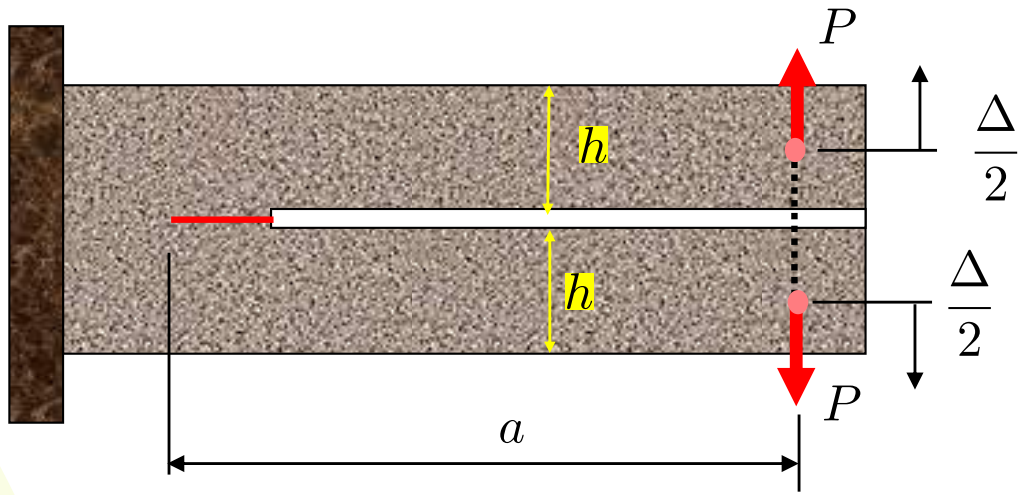
$$J = \frac{12P^2a^2}{B^2h^3E}$$

As the crack grows in length, how does the energy release rate (*J*) change at fixed load?

$$\left.\frac{\partial J}{\partial a}\right|_P = \frac{24P^2a}{B^2h^3E} = \frac{2J}{a} > 0$$

The rate of change of energy release rate is *positive*

Crack stability (2)



Thickness: B

$$J = \frac{12P^2a^2}{B^2h^3E}$$

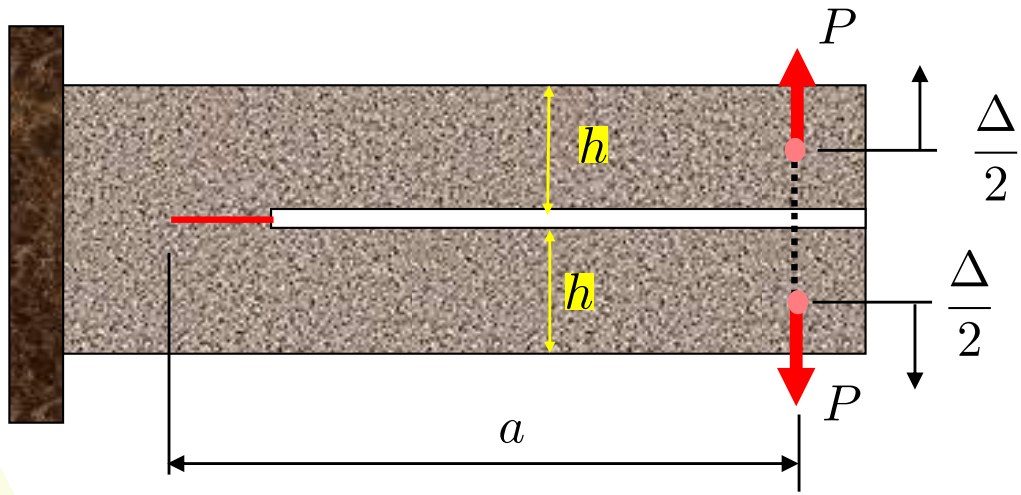
Change to displacement control loading

$$\Delta = \frac{2Pa^3}{3EI} \Rightarrow P = \frac{3EI\Delta}{2a^3}$$

$$J = \frac{9EI\Delta^2}{4a^4B}$$

Now compute rate of change of energy release rate at fixed, imposed displacement

Crack stability (3)



Thickness: B

$$\left.\frac{\partial J}{\partial a}\right|_P = \frac{2J}{a} > 0$$

Load control

$$\left.\frac{\partial J}{\partial a}\right|_\Delta = -\frac{J}{a} < 0$$

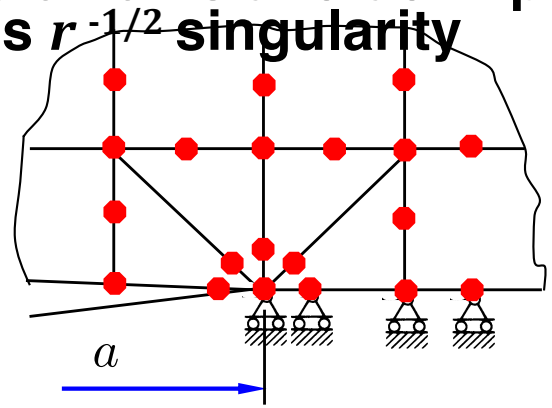
Displacement control

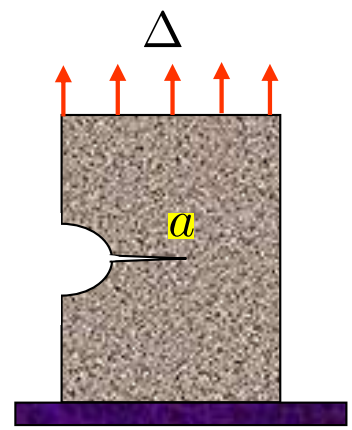
- **Crack growth begins when $J=J_{critical}$ for both cases**
- **Suppose the $J_{critical}$ is a material constant**
- **Then crack growth continues in load control since J will be above $J_{critical}$**
- **In displacement control, the crack stops since J decreases with growth. Additional imposed displacement is required to resume crack growth**
- **This also means that K_I increases in load control and decreases in displacement control during crack extension**

Use of FEA to compute J-K_I

- The finite element method is ideally suited to compute the energy release rate, J, and then K_I from the $J - K_I$ relationship
- Illustrate using displacement control loading of the FE model
- Real crack length is a, located at edge of some shape of hole
- Mode I conditions, model only upper ½ of specimen
- Use collapsed, 8-node 2-D elements at crack tip to best represent the strain-stress $r^{-1/2}$ singularity

$$\pi = U + \cancelto{0}{\Omega} = U = \frac{1}{2}\{u_s\}^T [K_s] \{u_s\}$$

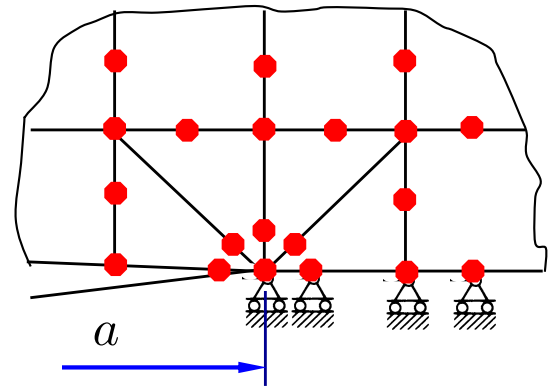




¼ point singularity elements. See Anderson Appendix 12

Use of FEA to compute J-K_I

$$\pi = U + \Omega = U = \frac{1}{2}\{u_s\}^T [K_s] \{u_s\}$$



$$a_1 = a - \Delta a \; ; \quad a_2 = a + \Delta a$$

$$\Delta a \approx 0.02a$$

- **Run 2 analyses with slightly longer and shorter crack lengths: just perturb the horizontal position of the 4 singularity elements**
- **Compute the strain energy, U, in each case. Most FEA codes print U if asked.**

$$J_{symm} = -\frac{\partial \pi}{\partial A_c} = -\frac{\partial U}{\partial A_c} = -\frac{1}{B}\frac{\partial U}{\partial a}$$

$$J_{symm} = -\frac{1}{B}\left[\frac{U(a+\Delta a) - U(a-\Delta a)}{2\Delta a}\right]$$

$$J = J_{total} = 2 \times J_{symm}$$

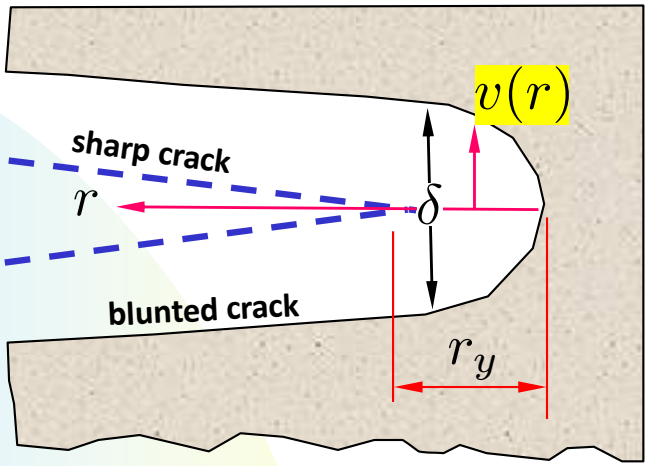
central difference estimate of derivative

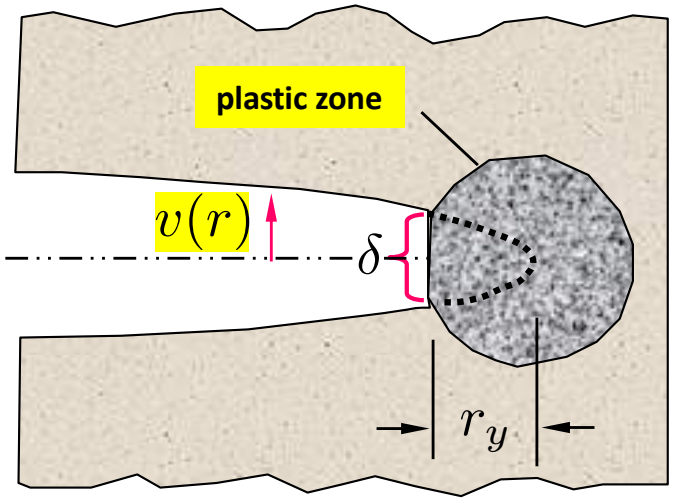
$$J > 0 \quad \text{since } U_{a-\Delta a} > U_{a+\Delta a}$$

(the FE model has only ½ of the specimen!)

$$K_I = \sqrt{E' J}$$

CTOD estimation





$$\underbrace{v(r) = \frac{4}{\pi} K_I \sqrt{\frac{r}{2\pi}}}_{\text{Plane Stress}} \Rightarrow \delta = 2 \times v(r = r_y) = \frac{8}{E} K_I \sqrt{\frac{\frac{1}{2\pi}\left(\frac{K_I}{\sigma_0}\right)^2}{2\pi}}$$

$$\therefore \delta = \frac{4}{\pi} \frac{K_I^2}{\sigma_0 E} \Rightarrow$$

CTOD increases as K^2
(this is a nonlinear effect!)

The actual coefficient (not 4/π) is found from nonlinear FE analyses. Waries with material strain hardening and 3-D effects.

Suggestion: re-derive for plane-strain conditions, verify physical units are correct.